MATHEMATICS MAGAZINE

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MATHEMATICS MAGAZINE

STEPHEN A. JENNINGS, *EDITOR*

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IN MEMORIAM SAMUEL T. SANDERS 1872–1970

Professor Samuel T. Sanders, founder and editor emeritus of MATHEMATICS MAGAZINE and one of the individuals principally instrumental in the organization in the Deep South of the Mathematical Association of America and the National Council of Teachers of Mathematics, died in Mobile, Alabama, March 19, 1970, after a short illness, at the age of ninety-eight.

Samuel Thomas Sanders was born in Washington County, Florida, on January 17, 1872, the son of George Whitfield Sanders and Lucinda Susan Willeford. He received the B.A. degree at Southern University (Birmingham-Southern College) in 1891. After graduation he taught school in the Alabama towns of Hope Hull and Snowdoun, and for two to three years he served as a part-time Methodist clergyman in Hope Hull and Elba, until 1898, when, anxious to contribute to the wartime service of his country, he enlisted in the 2nd Georgia Regiment, U. S. Volunteers, Spanish American War.

On December 1, 1898 he opened University High School, Baton Rouge, Louisiana. Four years later he became head of the subfreshman department of Louisiana State University. In 1907 he joined the University mathematics faculty, and in 1917 he succeeded Professor James W. Nicholson as Chairman of the Mathematics Department. In 1927 he received the M.S. degree at the University of Chicago. He retired as Professor Emeritus at L. S. U. in 1942. Colleagues who worked with him have written that "he had a depth of character and breadth of wisdom rare among college professors," "he was a fine friend and a good administrator," and "he strengthened the faculty and developed the graduate department at L. S. U. until it offered a strong and respected Ph.D. program." After taking up residence in Mobile in 1945 he did part-time teaching at the Mobile Center, University of Alabama, and later at the University Military School. He retired from teaching in 1957.

In 1923 Professor Sanders was selected by the Mathematical Association of America to represent Louisiana on a committee to act in a nation-wide campaign for more members. Through his efforts some college mathematics teachers from Louisiana and Mississippi met in Baton Rouge in February 1924, and plans were formulated for a program meeting and election of officers the following month. The group's petition for affiliation with the MAA was approved within four months and the Louisiana-Mississippi Section was chartered.

As third chairman of that Association Section, Professor Sanders began publishing, in 1926, the *Mathematics News Letter*, four-page circulars sent to secondary school as well as college teachers in the two states, urging attendance at the next meeting of the Section, in Shreveport, Louisiana. At that meeting, March 5–6, 1927, over ninety persons assembled, including Professor H. E. Slaught, representing the MAA and Miss Marie Gugle, president of the National Council of Teachers of Mathematics. There was formed the Louisiana-Mississippi Branch of the NCTM, the third branch to be affiliated with that national organization which is celebrating its Golden Jubilee this year [1]. Professor Sanders was re-elected chairman of the Association Section for the following year. He served the Association as a Governor from 1941–1943.

Lunching together immediately after that 1927 meeting, Professors Slaught, Sanders and P. K. Smith, Secretary-Treasurer of the Section, decided to continue the *Mathematics News Letter*, the first permanent mathematics journal to originate in the Deep South. Presently it declared itself to be "a journal dedicated to mathematics in general and to the following aims in particular: 1. A study of the common problems of secondary and collegiate mathematics teaching; 2. A true valuation of the disciplines of mathematics; 3. The publication of high class expository papers on mathematics; 4. The development of greater public interest in mathematics by the publication of authoritative papers treating its cultural, humanistic and historical phases." One has only to page through the early numbers of the publication to see how admirably these objectives were initiated under the editorship of Professor Sanders.

The *Mathematics News Letter* continued under this title through volumes I–VIII (1926–1934) and then in a gradually improved and enlarged form it appeared as the *National Mathematics Magazine* (IX–XX, 1934–1945).

Professor Sanders was the author of several expository monographs on mathematics published in the *L.S.U. Bulletin* and elsewhere, and from time to time he contributed expository articles to the journal, but for the most part he appeared in his numerous editorials, speaking out for the dignity of the mathematical profession, with concern for secondary school mathematics curricula, of the need of motivation and inspiration in the education process, on the utilization of "carried-over values" in learning, the promotion of cooperation between secondary school and college teachers, the importance of scholarship in college faculties and the value of an appreciation of the history of mathematics and of the philosophy of mathematics.

Although the journal soon had subscribers in almost every state of the Union it was in financial difficulties from its beginning. That it should succeed at all is a great tribute to the men who guided and sustained it through so many arduous years. In 1935 the journal was officially sponsored by L.S.U., until 1942, when on Professor Sanders' retirement, it was returned to his personal sponsorship. He was deeply saddened by this action although he understood and appreciated the priorities of the University in its decision. He immediately set about trying to find a Southern sponsor for the journal and to pay off its debt. Failing to find a place in the South for the journal was a great disappointment to him and he was hesitant to accept the offer of Professor Glenn James of the University of California, Los Angeles, in 1947, to take over the sponsorship and management. Shortly after that acceptance the mathematics department of the University of Alabama wrote to Professor Sanders expressing a willingness to assume responsibility for the publication, but he stood by his understanding with Professor James. To those who came to his financial aid, Professor Sanders was very grateful. Mention should be made of his son, Dr. S. T. Sanders, Jr., a mathematician of considerable promise, who met death April 10, 1945 from an automobile accident. Professor and Mrs. S. T. Sanders, Jr. contributed some $700 to pay off the debts of the journal.

Volume XXI (1947–1948) and succeeding volumes have had the title MATHEMATICS MAGAZINE. Professor Sanders continued to read the journal and follow its fortunes. For years he alluded to the misfortune that the management of the journal could not remain in the South, but little by little the realization that "a good and knowing Providence guided my way" grew upon him. He was deeply grateful for the very great personal sacrifices that were made by Professor James and his family over a period of twelve years, to finance and improve the journal [2, 3], and for the financial support that the MAA began to contribute in 1959 by an appropriation from the Jacob Houck Memorial Fund. The journal became an official publication of the MAA with volume XXXV (1962) [4].

In his retirement in Mobile, Professor Sanders wrote "Unstable Foundations of a Marxist State," a philosophical probing in a lucid style reminiscent of some of the better features of Spinoza's "geometrical" argumentation. This book-length manuscript was finished at a time when a "soft line" on Communism was becoming fashionable and the work remains unpublished.

References

1. H. E. Slaught, The Association and its sections, Amer. Math. Monthly, 34 (1927) 225–229.
2. Editorial acknowledgment, this MAGAZINE, 33 (1959–1960) facing p. 119.
3. D. H. Hyers, Glenn James 1882–1961, this MAGAZINE, 34 (1960–1961) facing p. 311.
4. R. E. Horton, Editorials facing pp. 125, 249, Ibid.

T. F. MULCRONE, S.J., Spring Hill College, Mobile

DYNAMIC PROOFS OF EUCLIDEAN THEOREMS

ROSS L. FINNEY, University of Illinois, Urbana

Simple observations about transformations of the plane lead to elegant proofs of unusual Euclidean theorems. The theorems are easily stated, and many of them can be conjectured from hand-drawn pictures. The proofs use neither coordinates nor vectors, and the simplest of them require only a slight familiarity with rotations and translations. We look at some of these first, and then introduce similarity transformations to show that a number of theorems which have appeared in the literature as isolated results are really special cases of two or

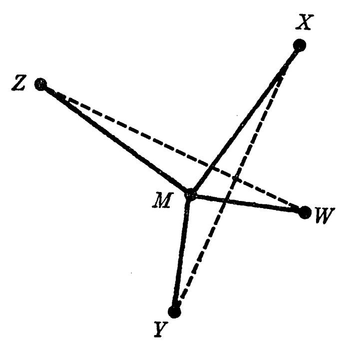

FIG. 1.

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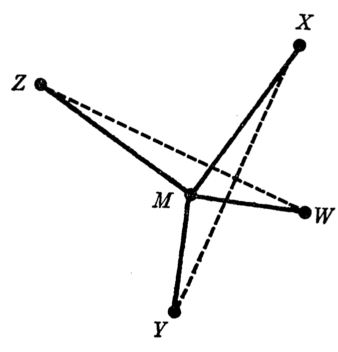

FIG. 1.

three general theorems. A pleasant by-product of the proofs given here is that they yield results about quadrilaterals without assuming them to be either convex or simple (compare, for example, Theorems 1 and 3 with their counterparts in [5]).

The first lemma is about Figure 1.

LEMMA 1. *If isosceles triangles ZMX and YMW have right angles at M, then $\overline{YX}$ and $\overline{ZW}$ are perpendicular and congruent.*

Proof. This is because M_{90}, the counterclockwise rotation of 90 degrees about M, moves Y to W and X to Z.

LEMMA 2. *If Z and X are centers of squares that lie on sides of ABC, built towards the exterior of ABC, and if M is the midpoint of the third side, then ZMX is isosceles and has a right angle at M.*

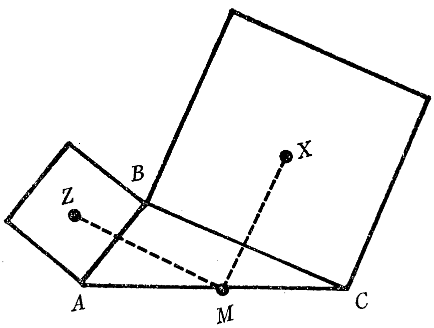

FIG. 2.

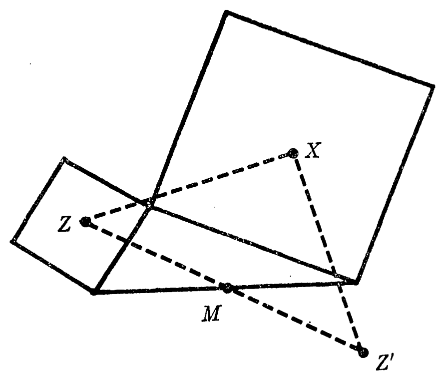

FIG. 3.

To see why, note that the composite $T = M_{180}X_{90}Z_{90}$ of rotations about Z, X and M is a translation. This is because the degree measures of the rotations add up to an integral multiple of 360. Under this composite, A goes first to B, then to C and finally back to A. But the only translation with a fixed point is the identity transformation. Since $T = I$, the successive images of Z are

$$Z_{90}(Z) = Z,$$

$$X_{90}(Z) = \text{some point which we call } Z',$$

and

$$M_{180}(Z') = T(Z) = Z.$$

Our picture now looks like Figure 3. Because $X_{90}(Z) = Z'$, we know that $ZX = XZ'$ and that ZXZ' has a right angle at X. We also know that M is the midpoint of $\overline{ZZ'}$ because $M_{180}(Z') = Z$. Thus M is the midpoint of the hypotenuse of a right isosceles triangle, which makes ZMX isosceles with a right angle at M.

The conclusion of Lemma 2 holds if the squares are constructed towards the interior of the triangle instead of towards the exterior. In the proof, one merely replaces the counterclockwise Z_{90} and X_{90} by the clockwise Z_{-90} and X_{-90}.

THEOREM 1. [1, 3] *If X, Y and Z, W are opposite pairs of centers of squares on the sides of a quadrilateral that lie towards the quadrilateral's exterior, then* $\overline{YX}$ *and* $\overline{WZ}$ *are perpendicular and congruent.*

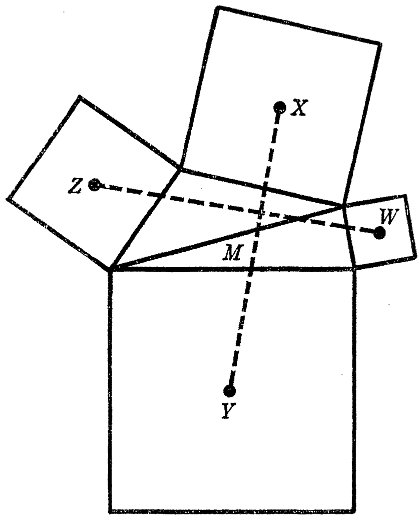

FIG. 4.

Proof. Let M be the midpoint of a diagonal of the quadrilateral, and apply M_{90}.

Theorem 1 holds also if the squares all lie towards the quadrilateral's interior. The proof is the same: M_{90} throws one segment onto the other.

If the quadrilateral happens to be a parallelogram, then $ZXWY$ is a square, because all four of ZMX, XMW, WMY and YMZ are isosceles and have right angles at M.

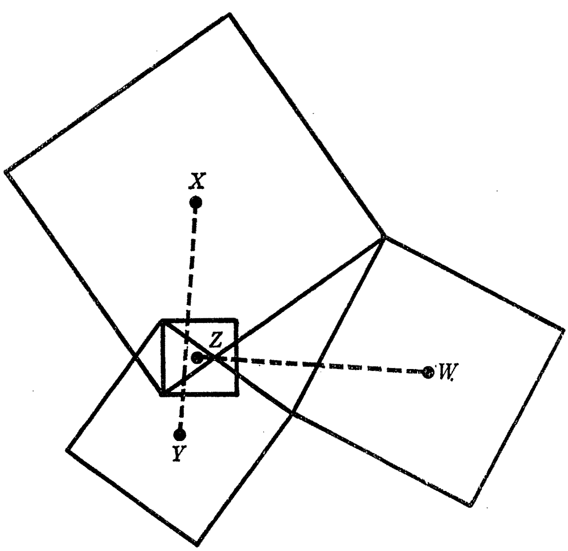

FIG. 5.

It should be emphasized here that the quadrilateral need not be convex, or even simple. To place the squares when the quadrilateral has no obvious interior, one traverses the quadrilateral in one of the two possible directions, laying off squares to the right.

If we shrink an edge of the quadrilateral to a point, we see a theorem about squares on a triangle.

THEOREM 2. [1, 3] *If squares are constructed on the sides of a triangle towards the triangle's exterior, then the segment joining two of the centers is perpendicular and congruent to the segment joining the third center to the vertex opposite it.*

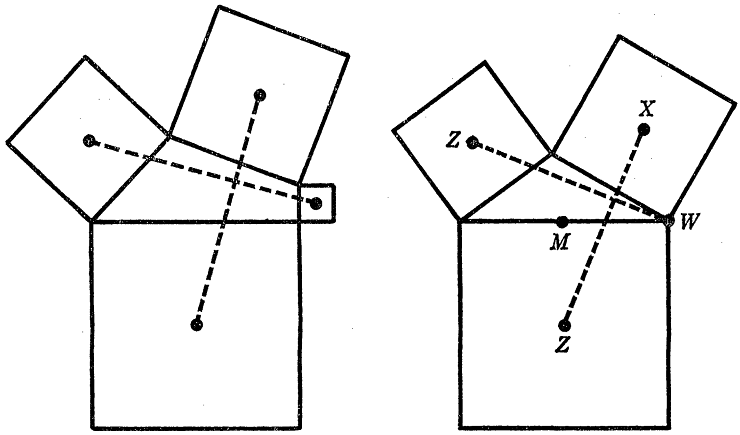

FIG. 6.

Proof. Look at Figure 6, and apply M_{90} as in Lemma 1.

Theorem 2 has the corollary that the lines joining the square centers to the vertices opposite them are concurrent (Figure 7). This is because the lines are altitudes of XYZ.

One of the first theorems of Euclidean geometry is that the midpoints of the sides of a quadrilateral are themselves the vertices of a parallelogram.

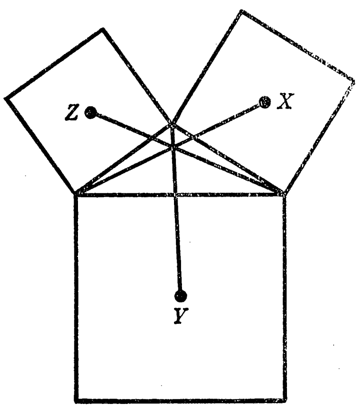

FIG. 7.

THEOREM 3. *The vertices Z, X, W, Y of equilateral triangles built on the sides of a quadrilateral, and lying alternately towards the interior and the exterior of the quadrilateral, are themselves the vertices of a parallelogram.*

Proof. The composite $C_{-60}A_{60}$ is a translation that takes Y to W and Z to X. Thus $\overline{YZ}$ and $\overline{WX}$ are parallel and congruent.

The Euclidean midpoint theorem and Theorem 3 are two of a family of theorems.

THEOREM 4. [2] *If Z, X, W, Y are vertices of similar triangles appropriately arranged on the sides of a quadrilateral, then $ZXWY$ is a parallelogram.*

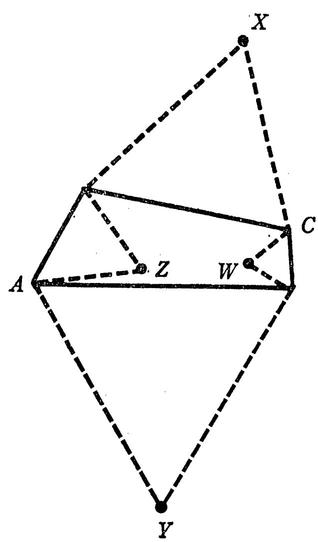

FIG. 8.

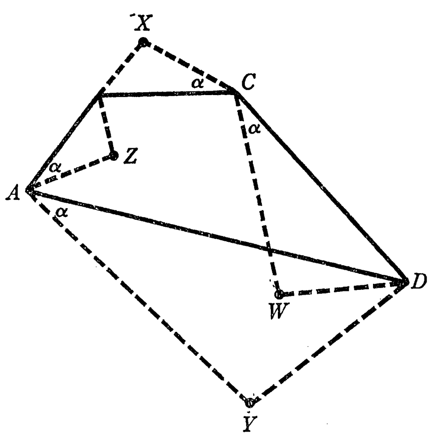

FIG. 9.

Proof. Let α be the measure of the angles that the four similar triangles have at A and C, and let r denote the ratio of AY to AD. Let $A^{1/r}$ denote the central dilatation with center A that multiplies distance by $1/r$. Let C^r denote the central dilatation with center C that multiplies distance by r. Then the composite $C_{-\alpha}C^rA^{1/r}A_\alpha$ is a translation that takes Y to W and Z to X.

If $\alpha=0$ we have a slight generalization of the midpoint theorem. If $r=1$ and $\alpha=60$, we have Theorem 3. Once again, the quadrilateral need not be convex or simple. Shrinking an edge leads to several nice theorems about triangles.

With similarity transformations, one can prove a theorem (Theorem 5) about triangles that has truly surprising corollaries. Among them are the following three:

COROLLARY 1. *Suppose that* 30–60–90 *triangles are built on two sides of an arbitrary triangle towards its exterior, as in Figure* 10. *Let Z and X denote the outer vertices of these triangles, and let M be the midpoint of the remaining side of the given triangle. Then ZMX is equilateral. If, instead, the* 30–60–90 *triangles lie toward the interior of the given triangle, ZMX is still equilateral.*

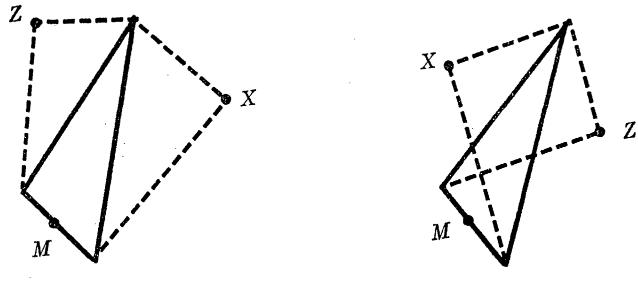

FIG. 10.

COROLLARY 2. (Napoleon's Theorem) *The centers X, Z, M of equilateral triangles constructed on the sides of an arbitrary triangle, and lying towards the triangle's exterior, are themselves the vertices of an equilateral triangle.*

COROLLARY 3. [5, Exercise 23] *Suppose that equilateral triangles are built on the sides of an arbitrary triangle, two towards its exterior and one towards its interior. Let M be the center of the inner one, and Z and X the apexes of the outer ones. Then ZMX is an isosceles triangle, with a $120°$ angle at M.*

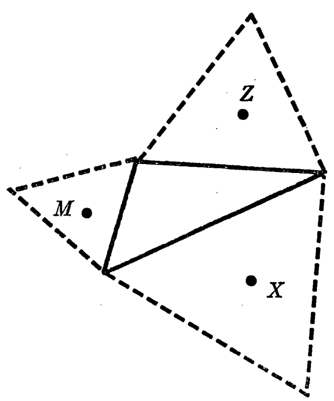

FIG. 11.

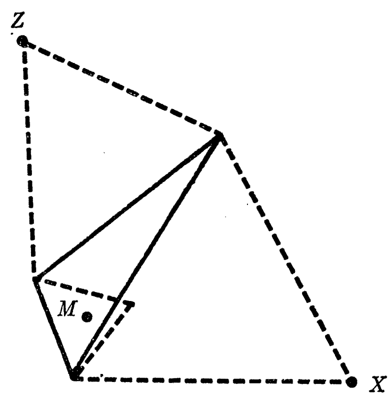

FIG. 12.

In order to state Theorem 5 simply, we assume that angles are "general," that is, that they have a variety of measures which agree modulo 360, and that they are oriented in a counterclockwise fashion. Thus, Figure 13 shows two angles: angle CBA of measure 60, 420, etc., and angle ABC of measure 300, -60, etc.

THEOREM 5. *Let BZC and CXA be nondegenerate similar triangles (with vertices corresponding in the order given) constructed both towards the exterior or both towards the interior of arbitrary triangle ABC. Let the angles BZC and CXA have measure β. Let M be the point in the plane that is equidistant from A and B and that is located so that angle BMA has measure 2β. Then $MZ = MX$.*

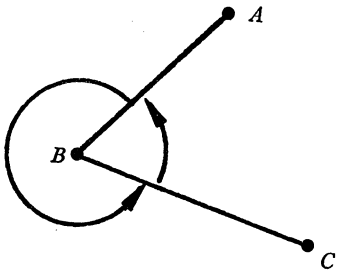

FIG. 13.

Proof. Since BZC and CXA are nondegenerate, $2\beta \neq 360$ and M really does exist. The triangle AMB may be degenerate, however, for if $\beta = 90$ then M is the midpoint of $\overline{AB}$.

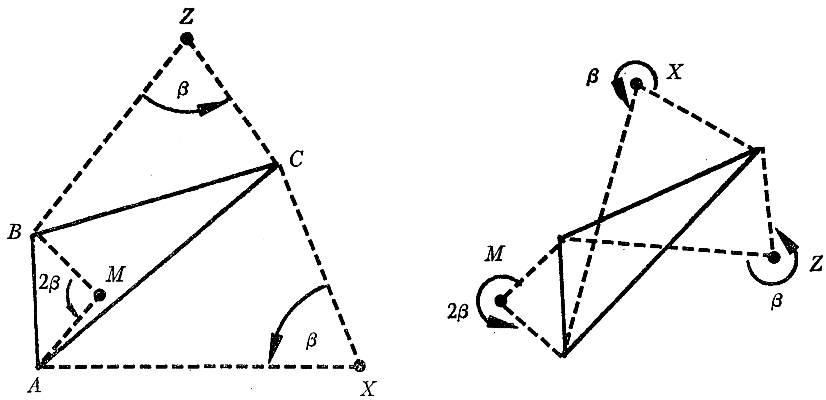

FIG. 14.

Let r denote the ratio ZC/BZ. Then the translation $T = X_\beta X^{1/r} Z^r Z_\beta M_{-2\beta}$ is the identity because it leaves A fixed. If we follow M about, we see that $M_{-2\beta}(M) = M$, that $Z^r Z_\beta(M)$ is some point M', say, and that $X_\beta X^{1/r}(M') = T(M) = M$. Thus $MXM'Z$ is a quadrilateral with $XM' = r \cdot MX$, $ZM' = r \cdot MZ$ and with congruent angles at X and Z. This means that triangles MXM' and MZM' are similar. But they have a common side, so they are congruent and $MZ = MX$.

Proofs of the corollaries. If $\beta = 90$ and $r = \sqrt{3}$, then MXM' and MZM' are 30–60–90 triangles. In particular, $MXM'Z$ has a 60 degree angle at M. This makes ZMX equilateral, which proves Corollary 1.

If $\beta = 120$ and $r = 1$, then Z and X are indistinguishable from M. Consequently $MZ = MX = XZ$, which is Napoleon's theorem.

Corollary 3 is the case $\beta = 60$ and $r = 1$.

If $\beta = -60$ and $r = 1$ we have Corollary 3 with two inner triangles and one outer.

The assignment $\beta = -120$, $r = 1$ produces Napoleon's theorem with three inward triangles.

Lemma 2 is also a corollary of Theorem 5, the squares lying toward the exterior if $\beta = 90$, $r = 1$, and towards the interior of $\beta = -90$, $r = 1$.

We close with a generalization of Lemma 2 that leads to a generalization of Theorem 1.

THEOREM 6. [4] *Suppose that similar isosceles triangles are constructed on the sides of an arbitrary triangle ABC, either both towards the interior of ABC or both towards the exterior. Let X be the apex of one, Z the orthocenter of the other, and let M be the midpoint of the remaining side of ABC. Let β be the measure of the angle at X. Then ZMX has a right angle at M and an angle of measure $\beta/2$ at X.*

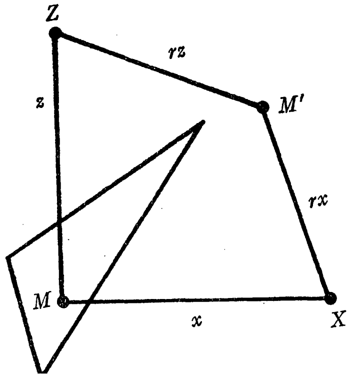

FIG. 15.

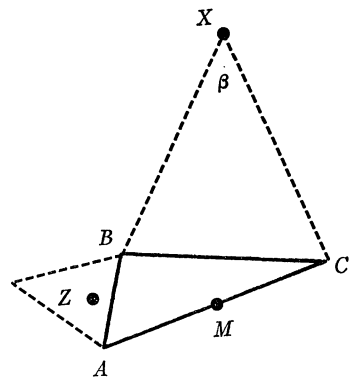

FIG. 16.

The proof is straightforward once one observes that angle AZB has measure $180\text{-}\beta$. Lemma 2 is the case $\beta=90$. If we let h denote the height of X above $\overline{BC}$, then $XM/MZ=BC/2h$. We call this latter number r in the next theorem.

THEOREM 7. *Suppose that similar isosceles triangles lie on the sides of a quadrilateral, all towards the interior or all towards the exterior of the quadrilateral. Let X and Y be apexes of one opposite pair of isosceles triangles, and let Z and W be*

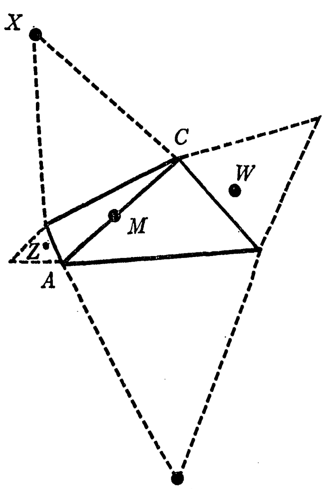

FIG. 17.

the orthocenters of the other two. Then $\overline{YX}$ and $\overline{WZ}$ are perpendicular, and $YX/WZ = r$.

Proof. Let M be the midpoint of the diagonal $\overline{AC}$. If the triangles lie towards the quadrilateral's exterior, then M^rM_{90} takes X to Z and Y to W. If the triangles lie toward the interior, apply $M^{1/r}M_{90}$ instead.

Again, the quadrilateral need not be convex or simple, and there is an analogous theorem for triangles.

Acknowledgements: I first saw dynamic proofs of Corollaries 1 and 2 in a course that Nicolaas Kuiper gave at the University of Michigan in 1953. Later that year Kenneth Leisenring showed me Theorem 3, and in 1965 he encouraged me to develop dynamic proofs for the African Mathematics Secondary Program. I am grateful to them.

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ANSWERS

A482. For any reals α, β the function

$$f(x) = \left(\sum_{i=1}^{M} x_i^{x-\alpha}\right)\left(\sum_{i=1}^{M} a_i^{\beta-x}\right)$$

is convex and symmetric about $x = 1/2(\alpha+\beta)$, and so increases as x moves away from $x = 1/2(\alpha+\beta)$. With $\alpha = 0$ and $\beta = P$ the given inequality is $f(P) \leqq f(P+1)$.

A483. Expanding out and replacing tan A, tan B and tan C by a, b, and c, respectively we get

$$\frac{a-b}{1+ab} + \frac{b-c}{1+bc} + \frac{c-a}{1+ca} = 0.$$

On combining fractions and factoring we obtain

$$(a-b)(b-c)(c-a) = 0$$

and thus the triangle is isosceles. Note the condition that $A+B+C=\pi$ is not necessary.

A484. Fermat's theorem states that if p is a prime and m is not divisible by p, then $m^{p-1} \equiv 1 \pmod{p}$. The relationship $a^2+b^2=c^2$ may be written as

$$(a^{3-1} - 1) + (b^{3-1} - 1) = c^2 - 2 = (c^{3-1} - 1) - 1.$$

If neither a nor b is divisible by 3, then each of the terms on the left-hand side of the equation is divisible by 3. But, whether c is divisible by 3 or not, the right-hand side of the equation leaves a remainder upon division by 3. Hence, for the equation to hold, one of a, b must be a multiple of 3.

A485. The difference of the half diagonals is $1-\sqrt{2}/2$, so each segment through the vertices is $2-\sqrt{2}$. At each corner the segments cut off from the sides are $(2-\sqrt{2})/\sqrt{2}$, so the remaining segment is $\sqrt{2}-2(2-\sqrt{2})/\sqrt{2}$ or $2-\sqrt{2}$. Each interior angle is 135° so the octagon is regular.

A486. An integer and its fifth power end in the same digit, hence (n^5-n) ends in zero and so is divisible by 2 and 5. Its factors are $(n-1)(n)(n+1)(n^2+1)$ of which one of the first three must be divisible by 3. So (n^5-n) must be divisible by $2\times 3\times 5$.

(Quickies on pp. 235–236.)

AN IMPLICATION OF THE PYTHAGOREAN THEOREM

JOHN Q. JORDAN and JOHN M. O'MALLEY, JR., Boston State College

It is well known that a Banach space in which the parallelogram law

$$\|x+y\|^2+\|x-y\|^2=2(\|x\|^2+\|y\|^2)$$

holds is a Hilbert space [3, p. 23]. This law is equivalent to the Pythagorean theorem. Also Hilbert space is a generalization of Euclidean space. Therefore it is natural to conjecture that the Pythagorean theorem implies Euclid's parallel postulate, or an equivalent proposition.

This conjecture is correct, but simple proofs of this fact do not seem to be well known. Two such proofs are presented in this note. The first is a straightforward application of the Pythagorean theorem in a general right triangle. The second proof is simpler, but rather artificial because it uses an isosceles right triangle rather than a general one.

The reader will recall that an absolute geometry is Euclidean if and only if there exists at least one triangle the sum of the measures of whose angles is 180 [1, p. 11]. We shall use this proposition rather than one of the more common forms (e.g., Playfair's) of Euclid's parallel postulate.

In absolute geometry if the Pythagorean theorem holds, it is trivially true that its converse is also valid.

After these preliminaries we can proceed with our proofs that the Pythagorean theorem implies Euclid's parallel postulate. We shall use two lemmas, one of which may be quite familiar to the reader.

LEMMA 1. *If a rectangle exists, then there is a triangle the sum of the measures of whose angles is* 180.

Proof. Given a rectangle $\square ABCD$. The sum of the measures of its angles is 360. Also the sum of the measures of these angles is the sum of the measures of the angles of $\triangle ABD$ and $\triangle BCD$. In absolute geometry we know [4, p. 130] that

$$0 < m(\angle BAD) + m(\angle ABD) + m(\angle ADB) \leqq 180$$

and

$$0 < m(\angle BCD) + m(\angle CBD) + m(\angle CDB) \leqq 180.$$

Hence, in both cases the equality must hold. This proves the lemma.

LEMMA 2. *Given a right triangle $\triangle ABC$ with the lengths $AB=c$, $AC=b$, $BC=a$ and $\angle ACB$ the right angle. Let h be the length of the altitude from C to $\overleftrightarrow{AB}$. If the Pythagorean theorem holds, then $h=ab/c$.*

Proof. Let D be the foot of the perpendicular from C to $\overleftrightarrow{AB}$. We know D is between A and B [4, p. 129]. Let $x=BD$. Then $AD=c-x$. By the Pythagorean theorem $a^2=h^2+x^2$ and $b^2=(c-x)^2+h^2$. Whence $2cx=a^2-b^2+c^2=2a^2$ and so $x=a^2/c$. Therefore $h^2=a^2-x^2=a^2(1-(a^2/c^2))$

$$h^2 = \left(\frac{a}{c}\right)^2 (c^2 - a^2), \quad \text{or} \quad h^2 = (ab)^2/c^2.$$

Hence $h=ab/c$.

THEOREM 1. *If the Pythagorean theorem holds, then a rectangle exists.*

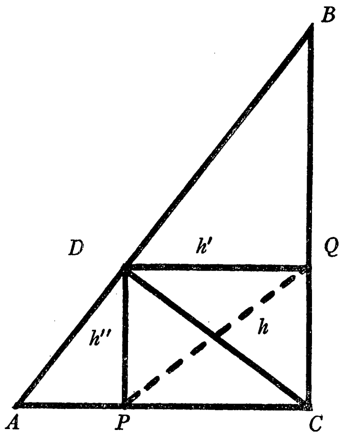

FIG. 1.

Proof. Given right triangle $\triangle ABC$ with $\angle C$ the right angle. Let $AB=c$, $BC=a$ and $AC=b$. Let D be the foot of the perpendicular from C to $\overleftrightarrow{AB}$, Q the foot of that from D to $\overleftrightarrow{BC}$ and P the foot of that from D to $\overleftrightarrow{AC}$. We know D is between A and B, P between A and C and Q between B and C. Let $h=DC$, $h'=DQ$ and $h''=DP$. The situation is shown in Figure 1.

By Lemma 2, $h=ab/c$. $\triangle ADC$ is a right triangle so $(AD)^2+h^2=b^2$. Hence

$$(AD)^2 = b^2 - \frac{a^2b^2}{c^2} = b^2\left(\frac{b}{c}\right)^2$$

and

$$AD = b\left(\frac{b}{c}\right), \qquad BD = c - \frac{b^2}{c} = a\left(\frac{a}{c}\right).$$

Since $\triangle DBC$ is a right triangle, then by Lemma 2

$$h' = h\left(\frac{a^2}{c}\right)\left(\frac{1}{a}\right) = \frac{ha}{c} = b\left(\frac{a}{c}\right)^2.$$

$\triangle DBQ$ is a right triangle, so $(h')^2+(BQ)^2=(BD)^2$. Therefore

$$(BQ)^2 = \frac{a^4}{c^2} - (h')^2 = \frac{a^4}{c^2} - \frac{b^2a^4}{c^4} = \frac{a^6}{c^4}$$

and so $BQ=a^3/c^2$. Consequently $QC=a-a^3/c^2=a(b/c)^2$.

By Lemma 2 since $\triangle ADC$ is a right triangle

$$h'' = h\left(\frac{b^2}{c}\right)\frac{1}{b} = a\left(\frac{b}{c}\right)^2 \qquad \text{and so } h'' = QC.$$

Since $\triangle DPC$ is a right triangle,

$$(PC)^2 = h^2 - h''^2 = \frac{a^2b^2}{c^2} - \frac{a^2b^4}{c^4} = \frac{a^2b^2}{c^2}\left(1 - \frac{b^2}{c^2}\right) = b^2\left(\frac{a}{c}\right)^4.$$

Therefore $PC=b(a/c)^2=h'=DQ$. Morevoer $h^2=h'^2+h''^2$.

In quadrilateral $\square PDQC$, $\angle P$, $\angle Q$, and $\angle C$ are right angles. Consider $\triangle DPC$ and $\triangle QCP$. $\angle P \cong \angle C$, $DP=h''=QC$ and $PC=PC$. Hence $\triangle DPC \cong \triangle QCP$ and so $PQ=h$. Consequently $(PQ)^2=h'^2+h''^2=(DQ)^2+(DP)^2$. Therefore $\triangle DPQ$ is a right triangle and $\angle D$ is a right angle. Hence $\square PDQC$ is a rectangle. Therefore, by Lemma 1 there is a triangle the sum of the measures of whose angles is 180, which proves the geometry is Euclidean.

THEOREM 2. *If the Pythagorean theorem holds, then the sum of the measures of the angles of an isosceles right triangle is* 180.

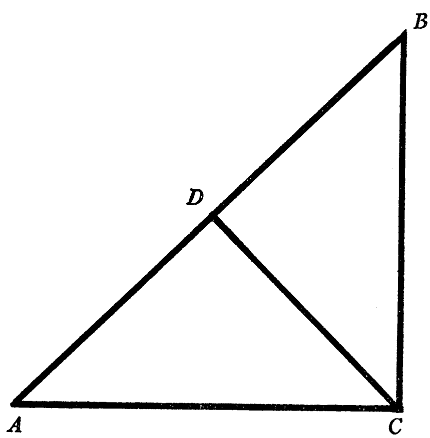

FIG. 2.

Proof. Let $\triangle ABC$ be an isosceles right triangle with $\angle C$ the right angle as in Figure 2. Also let $AB=c$, $BC=a$ and $AC=b$.

The bisector of $\angle C$ meets $\overline{AB}$ in some interior point D [4, p. 69]. Then $\overline{AC}\cong\overline{BC}$, $\overline{DC}\cong\overline{DC}$ and $\angle ACD\cong\angle BCD$. Consequently $\triangle ADC\cong\triangle BDC$ and so $\overline{AD}\cong\overline{DB}$. Therefore $AD=DB=c/2$. $\angle ADC\cong\angle BDC$, so both are right angles. By the Pythagorean theorem then $b^2=c^2/4+(DC)^2$ and $c^2=a^2+b^2$. Since $b=a$, then $c^2=2b^2$. Therefore $c^2/2=c^2/4+(DC)^2$ or $DC=c/2$. But also $AD=DB=c/2$, so $\triangle ADC$ and $\triangle BDC$ are both isosceles right triangles. From this it follows that $\angle A\cong\angle ACD\cong\angle B\cong\angle BCD$. Then since $m(\angle ACD)+m(\angle BCD)=90$, it follows that $m(\angle A)+m(\angle B)=90$. Hence $m(\angle A)+m(\angle B)+m(\angle C)=180$. Thus the sum of the measures of the angles of $\triangle ABC$ is 180, which proves the geometry is Euclidean.

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THE NUMBER OF SEGMENTS NEEDED TO EXTEND A CUBE TO *N* DIMENSIONS

ADRIEN L. HESS and CARL DIEKHANS, Montana State University

Any line segment may be regarded as being made up of two segments of lengths a and b. For sake of brevity the segments will be called a and b. The length of the original segment can then be represented by $(a+b)$. For convenience, let a be greater than b. On this segment let a square be constructed, each of whose sides is of length $(a+b)$. If the points common to the two segments be joined with a segment for each pair of parallel sides the original square region will be separated into two square regions and two rectangular regions. The two squares are a on a side and b on a side respectively. The two rectangles are each a by b. From the binomial theorem

$$(a+b)(a+b) = a^2 + 2ab + b^2.$$

This is a representation of a square in two dimensions whose sides are $(a+b)$ in length. The process above will be called the extension of a one dimensional representation to a two dimensional representation.

On this square let a cube be constructed each of whose sides is $(a+b)$. With the addition of the segments joining the points common to a and b to the respective parallel faces the cube is composed of eight parts—two cubes a^3 and b^3 respectively and three parallepipeds each of a^2b and ab^2. By the binomial theorem

Proof. Let $\triangle ABC$ be an isosceles right triangle with $\angle C$ the right angle as in Figure 2. Also let $AB=c$, $BC=a$ and $AC=b$.

The bisector of $\angle C$ meets $\overline{AB}$ in some interior point D [4, p. 69]. Then $\overline{AC}\cong\overline{BC}$, $\overline{DC}\cong\overline{DC}$ and $\angle ACD\cong\angle BCD$. Consequently $\triangle ADC\cong\triangle BDC$ and so $\overline{AD}\cong\overline{DB}$. Therefore $AD=DB=c/2$. $\angle ADC\cong\angle BDC$, so both are right angles. By the Pythagorean theorem then $b^2=c^2/4+(DC)^2$ and $c^2=a^2+b^2$. Since $b=a$, then $c^2=2b^2$. Therefore $c^2/2=c^2/4+(DC)^2$ or $DC=c/2$. But also $AD=DB=c/2$, so $\triangle ADC$ and $\triangle BDC$ are both isosceles right triangles. From this it follows that $\angle A\cong\angle ACD\cong\angle B\cong\angle BCD$. Then since $m(\angle ACD)+m(\angle BCD)=90$, it follows that $m(\angle A)+m(\angle B)=90$. Hence $m(\angle A)+m(\angle B)+m(\angle C)=180$. Thus the sum of the measures of the angles of $\triangle ABC$ is 180, which proves the geometry is Euclidean.

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$$(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3.$$

This process will be called the extension of a two dimensional representation to a three dimensional representation.

In a similar manner a three dimensional representation can be extended to a four dimensional representation. The resulting representation is called a hypercube or a tesseract. The tesseract is composed of two hypercubes or simple tesseracts a^4 and b^4 respectively, four hyperprisms of the form a^3b, six hyperprisms of the form a^2b^2, and four hyperprisms of the form ab^3. From the binomial theorem.

$$(a + b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4.$$

As dimensions are added, more algebra is involved. It can be observed that in a square two lines meet in a point, in a cube three and in a tesseract four. Using the theory of combinations the number of line segments or edges needed to make each model can be determined. If the vertices, edges, faces, and so on are counted the numbers will be found to be the binomial coefficients.

Consider the expression $(2c+d)$ where the coefficient 2 on c indicates the number of vertices and the coefficient 1 on d indicates the number of edges. The number of vertices, edges, faces, and so on, can be determined for a simple representation of a square, a cube, a tesseract, and so on. $(2c+d)=2c+d$ indicates 2 vertices and 1 edge. $(2c+d)^2=4c^2+4cd+d^2$ indicates 4 vertices, 4 edges and 1 face. $(2c+d)^3=8c^3+12c^2d+6cd^2+d^3$ indicates 8 vertices, 12 edges, 6 faces and 1 cube. $(2c+d)^4=16c^4+32c^3d+24c^2d^2+8cd^3+d^4$ indicates 16 vertices, 32 edges, 24 faces, 8 cubes and 1 tesseract. From this it is clear that in a simple tesseract there are 32 edges.

Now consider the expression for

$$(a + b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4.$$

Using only the a's from the above expression on the right it becomes

$$a^4 + 4a^3 + 6a^2 + 4a.$$

a^4 is a simple tesseract and has 32 edges. For $4a^3$ there are $4\cdot 12=48$ edges. $6a^2$ has $6\cdot 4=24$ edges. $4a$ has $4\cdot 1=4$ edges. The sum

$$32 + 48 + 24 + 4 = 108,$$

which is the total number of edges. Since each edge of the original tesseract consists of an a segment and a b segment there will also be 108 b segments.

Consider now the Nth dimensional extension of a cube. The 2^N vertices will determine $N\cdot 2^{N-1}$ segments. (The proof is by mathematical induction; for $N=1$ there are $2^1=2$ vertices and $1\cdot 2^0=1$ edge.) For $N=2$ there are $2^2=4$ vertices and $2\cdot 2=4$ edges. By the induction hypothesis for $N=K$ for 2^K vertices there are $K\cdot 2^{K-1}$ edges. It must now be shown that when $N=K+1$ then for 2^{K+1} vertices there will be $(K+1)\cdot 2^K$ edges.

$$2^{K+1} = 2\cdot 2^K = 2^K + 2^K \text{ vertices}.$$

From the induction hypothesis for each 2^K vertices there are $K\cdot 2^{K-1}$ edges.

Thus 2^K+2^K vertices determine

$$K\cdot 2^{K-1} + K\cdot 2^{K-1} = 2K\cdot 2^{K-1} = K\cdot 2^K \text{ edges.}$$

In an extension from a given dimension to the next higher dimension one more edge is added at each of the 2^K vertices. Therefore, the extension from K dimensions to $(K+1)$ dimensions has $(K+1)$ edges meeting at each vertex. Therefore, the total number of edges of a $(K+1)$th dimension extension of a cube is

$$K(2^{K-1}) + K(2^{K-1}) + 1\cdot 2^K = 2K\cdot 2^{K-1} + 2^K = K\cdot 2^K + 2^K = (K+1)2^K.$$

Therefore, for 2^N vertices there are $N(2^{N-1})$ edges. If the Nth dimensional extension of a cube is represented by a^N where each edge is of length a, then there are $N\cdot 2^{N-1}$ a length segments.

Consider the cube $(a+b)^3$ which is to be extended to the Nth dimension. Using the binomial theorem one obtains

$$(a+b)^N = a^N + Na^{N-1}b + \frac{N(N-1)}{2}a^{N-2}b^2 + \cdots + \frac{N(N-1)}{2}a^2b^{N-2} + Nab^{N-1} + b^N.$$

Now considering just the powers of a in the right hand side one has

$$a^N + Na^{N-1} + \frac{N(N-1)}{2}a^{N-2} + \cdots + \frac{N(N-1)}{2}a^2 + Na.$$

The terms represent the extension of the cube a^3 to the indicated dimensions. By replacing the powers of a by the number of edges for each simple representation the result is

$$N(2^{N-1}) + N[(N-1)2^{N-2}] + \frac{N(N-1)}{2}[(N-2)2^{N-3}] + \cdots + \frac{N(N-1)}{2}[2\cdot 2^1] + N[1\cdot 2^0].$$

This can now be written as

$$N\left[2^{N-1} + (N-1)2^{N-2} + \frac{(N-1)(N-2)}{2}2^{N-3} + \cdots + (N-1)2 + 1\right].$$

But the part in the brackets is the binomial expansion of

$$(2+1)^{N-1} = 3^{N-1}.$$

Therefore, the total number of edges in the Nth dimensional extension of a cube is $N\cdot 3^{N-1}$. Since in the beginning each edge was considered to be made up of two segments a and b, then there are $N\cdot 3^{N-1}$ segments of length a and $N\cdot 3^{N-1}$ segments of length b.

It is now possible to calculate, theoretically, the number of feet of wire that is needed to construct a model of a tesseract or any higher dimensional model of finite number N.

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A GEOMETRIC PROOF OF THE NONEXISTENCE OF PG_7

STEVEN H. HEATH and C. R. WYLIE JR., Furman University, Greenville, S. C.

It is well known that there is no finite projective geometry with the property that each of its lines contains exactly seven points. This was first demonstrated in 1901 by Tarry [1] who used an argument based on the properties of latin squares to show that the equivalent "problem of the 36 officers" had no solution. Then in 1948, Bruck and Ryser [2] showed by a number theoretic argument the existence of an infinite sequence of values of n, beginning with $n=7$, for which no PG_n exists. Since finite geometries are often considered in some detail in courses in the foundations of geometry for teachers and teaching majors, a relatively simple and essentially geometric proof of the nonexistence of PG_7 may be of interest. Such a proof is given in this paper.

In summary, our argument consists of an attempted enumeration of the complete four-points in PG_7 which have two of their diagonal points fixed. Since it proves impossible to carry out this enumeration without contradiction, we are thus led to the conclusion that no PG_7 can exist.

We begin with several lemmas on the number of four-points in PG_7 which have a given triangle as diagonal triangle:

LEMMA 1. *Two four-points with the same diagonal triangle cannot have more than one vertex in common.*

LEMMA 2. *In PG_7 there are at most two four-points having a given triangle as diagonal triangle and having a vertex in common.*

Proof. Suppose, contrary to the lemma, that there are three four-points having a common vertex and a common diagonal triangle. Without loss of generality these may be taken to be the four-points (9, 11, 21, 23), (10, 11, 28, 29), and (11, 12, 17, 18) shown in Figure 1, with the vertex 11 in common and the common diagonal triangle (1, 7, 38). Then the points 7, 11, 18, 21, 28 must be collinear on a line l. Moreover, from the usual axioms for a finite projective geometry, l must intersect l_2. Since this is impossible, because l cannot have a second intersection with l_7, l_8, l_9, l_{10}, l_{11}, l_{12}, or l_{13}, the lemma is established.

LEMMA 3. *In PG_7, if two four-points have a vertex in common and a common diagonal triangle, no vertex of either four-point can be a vertex of a third four-point with the same diagonal triangle.*

Proof. Without loss of generality, the two four-points with a common vertex and a common diagonal triangle may be taken to be the four-points (11, 12, 17,

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LEMMA 3. *In PG_7, if two four-points have a vertex in common and a common diagonal triangle, no vertex of either four-point can be a vertex of a third four-point with the same diagonal triangle.*

Proof. Without loss of generality, the two four-points with a common vertex and a common diagonal triangle may be taken to be the four-points (11, 12, 17,

18) and (10, 11, 28, 29) in Figure 1. Suppose that another four-point having (1, 7, 38) as its diagonal triangle has a vertex, say 18, in common with one of these four-points. Neglecting those which can be immediately rejected, there are four possibilities for such a third four-point: (18, 14, 20, 24), (18, 15, 21, 24), (18, 15, 33, 36), and (18, 16, 28, 30). If the third four-point is (18, 14, 20, 24), it follows that the points 7, 11, 18, 20, 28 must be collinear on a line l. Furthermore, since l must have a point in common with l_2, it must also contain the point 33. It also follows that the points 7, 10, 29 are collinear on a line l'. Now l' must intersect l_2, l_4, l_5 and l_{10}, l_{13}, l_{14}. Moreover, since l' contains only six points besides its intersection with l_1, its intersections with l_2, l_4, l_5 must also be its intersections with l_{10}, l_{13}, l_{14}, in one order or another. Therefore, since the intersections of l_2 with l_{13} and l_{14} are 33 and 32, respectively, and since these are on l and l_8, respectively, which already have 7 in common with l', it follows that l' must intersect l_2 in the point common to l_2 and l_{10}, namely 36. This in turn implies that the intersection of l' and l_4 must be either the intersection of l_4 and l_{13}, namely 21, or the intersection of l_4 and l_{14}, namely 20. The latter is clearly impossible, for otherwise l' would have both 7 and 20 in common with l. Hence l' also contains

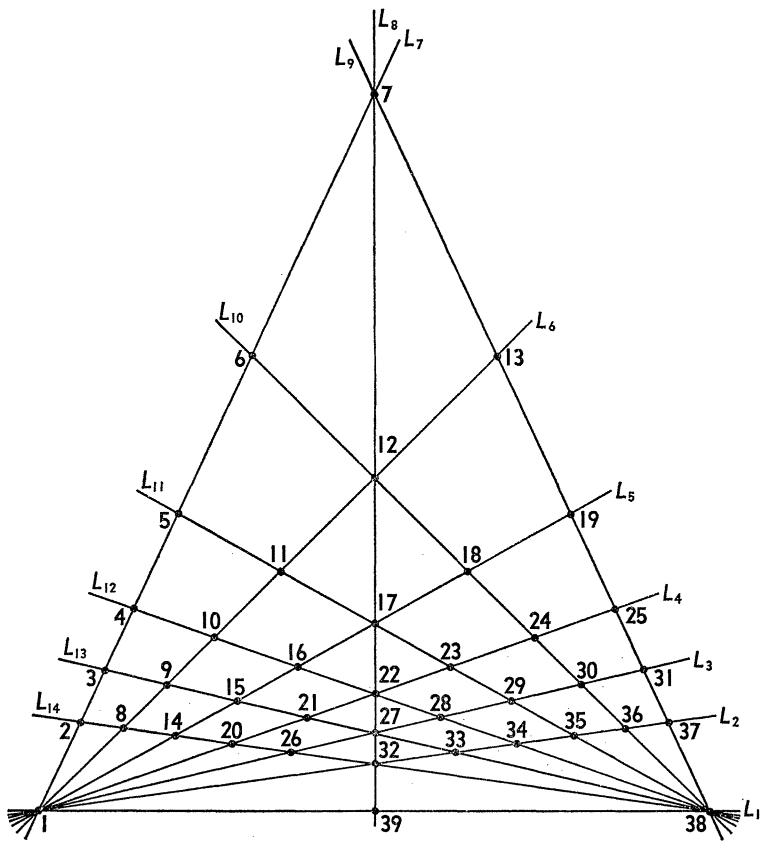

FIG. 1.

21 and, finally, the intersection of l_5 and l_{14}, namely 14. However, from the four-point (14, 18, 20, 24) it is clear that the line determined by 7 and 14 must also contain 24, which is impossible since it already has 21 in common with l_4. This contradiction forces us to reject the possibility that the four-point (18, 14, 20, 24) has the diagonal triangle (1, 7, 38). The four-points (18, 15, 21, 24) and (18, 15, 33, 36) are related to the original two four-points precisely as the four-point (18, 14, 20, 24) and can be rejected by an identical argument.

If the third four-point is (18, 16, 28, 30), then the points 7, 11, 18, 28 must be collinear on a line l. Furthermore, l must intersect l_2 and l_4 and these points, in one order or the other, must also be the points common to l and l_{13} and l_{14}. Clearly, l cannot contain the intersection of l_2 and l_{14}, namely 32, since this would force l and l_8 to have both 7 and 32 in common. Hence l must contain the intersection of l_2 and l_{13}, namely 33, and the intersection of l_4 and l_{14}, namely 20. Now 7, 10, 29 are also collinear on a line l'. Furthermore, l' must intersect l_2, l_4, l_5 and l_{10}, l_{13}, l_{14} and, precisely as in the discussion above, these intersections must be 36, 21, 14. Finally, the points 7, 16, 30 must be collinear on a line l'', and this line must intersect l_2, l_4, l_6 in points which in one order or another are the points common to l'' and l_{11}, l_{13}, l_{14}. Now the intersections of l_2 with l_{13} and l_{14} are 33 and 32, and the intersections of l_4 with l_{13} and l_{14} are 21 and 20. Moreover, none of these points can lie on l'' for otherwise l'' would have two points in common with at least one of the lines l_1, l', l_8. Hence the intersections of l'' with l_2 and l_4 must be the intersections of these lines with l_{11}. But this, too, is impossible since l'' and l_{11} cannot have two points in common. Hence the four-point (18, 16, 28, 30) cannot have (1, 7, 38) as its diagonal triangle.

In the same way, it can be shown that no four-point having either 12 or 17 as a vertex can have (1, 7, 38) as its diagonal triangle. Since the relation between the four-points (11, 12, 17, 18) and (10, 11, 28, 29) is completely symmetric, it is unnecessary to consider explicitly the possibility of a third four-point having 10, 28, or 29 as a vertex and (1, 7, 38) as its diagonal triangle. Hence the lemma is established.

LEMMA 4. *In PG_7 three four-points cannot have the same diagonal triangle if two of the four-points have a vertex in common.*

Proof. Consider two arbitrary four-points with a vertex in common and the same diagonal triangle, as the four-points (11, 12, 17, 18) and (10, 11, 28, 29) in Figure 1. As in the proofs of Lemmas 2 and 3, it follows that the points 7, 11, 18, 20, 28, 33 are collinear on a line l and the points 7, 10, 14, 21, 29, 36 are collinear on a line l'. Now if a third four-point has (1, 7, 38) as its diagonal triangle, then by Lemma 3 it can have no vertex in common with either (11, 12, 17, 18) or (10, 11, 28, 29). Moreover, no vertex of such a four-point can be a point of l, l', or l_8 unless the opposite vertex is also a point of that line, for otherwise there would be two lines containing both that vertex and the point 7. By inspection, it is clear that there are no four-points satisfying these restrictions. For example, (8, 9, 14, 15) is a four-point having no vertex in common with either (11, 12, 17, 18) or (10, 11, 28, 29), but it contains the point 14 which is on l' while the opposite vertex, 9, is not. Thus the lemma is established.

LEMMA 5. *In PG_7, three four-points such that no two have a vertex in common cannot have the same diagonal triangle.*

Proof. Suppose, contrary to the lemma, that there are three four-points with the same diagonal triangle such that no two of them have a vertex in common. Since two lines of each four-point pass through a given diagonal point and since there are only five lines on each diagonal point which can be lines of the four-points (because no vertex of a four-point can be collinear with two diagonal points), it follows that at least one line on each diagonal point is a line of at least two of the three four-points. If we focus our attention on the diagonal point 7 and let l_8 be the line which serves as a line of two of the four-points, then without loss of generality these four-points can be taken to be (11, 12, 17, 18) and (21, 22, 27, 28), as in Figure 1.

Let us suppose first that the four-points (11, 12, 17, 18) and (21, 22, 27, 28) have in common a second line, l, passing through 7. Then l must contain the points 7, 11, 18, 21, 28. Furthermore, l must intersect both l_2 and l_{14}, and since l contains only six points apart from its intersection with l_1, it must intersect l_2 and l_{14} in the same point, namely 32. But this is impossible, for then l would have both 7 and 32 in common with l_8. Hence the two four-points cannot have two lines on 7 in common.

The only other possibility is that the points 7, 11, 18 and 7, 21, 28 are collinear on distinct lines, say l and l', respectively. Then l must intersect l_2, l_3, l_4 and in one order or another these points must also be the intersections of l with l_{12}, l_{13}, l_{14}. Now the intersections of l_4 with l_{12} and l_{13} are 22 and 21, respectively, and the intersections of l_3 with l_{12} and l_{13} are 28 and 27, respectively. If l contained any one of these four points it would, perforce, have two points in common with either l' or l_8, which is impossible. Hence l must intersect l_3 and l_4 where these two lines intersect l_{14}. But this, too, is impossible, since l cannot have two points in common with l_{14}. Hence the lemma is established.

Lemmas 4 and 5 together establish the following result:

LEMMA 6. *In PG_7 there are at most two four-points with a common diagonal triangle.*

Let us now consider the set of all four-points in PG_7 which have two of their three diagonal points fixed, say at the points 1 and 38 in Figure 1. Clearly, the number of such four-points is $({}_6C_2)^2 = 225$. On the other hand, since there are only 36 points which are not on l_1, that is, since there are only 36 triangles in PG_7 which have points 1 and 38 as two of their vertices, it follows from Lemma 6 that at most 72 of the 225 four-points we have just noted can have noncollinear diagonal points. In other words, there must be at least 153 four-points which have 1 and 38 as diagonal points and have their third diagonal point collinear with 1 and 38.

Since there are five points on l_1 distinct from 1 and 38, at least one of these points, say the point 39 in Figure 1, must be the third diagonal point of at least 31 four-points. Collectively, these 31 four-points involve 124 vertices, counting duplications, of course. Since there are six lines distinct from l_1 on the point 39, and since each side of a four-point contains an even number of vertices, it follows further that at least one of these lines, say l_8, must contain at least 22 of these 124 vertices, that is, must be a side of at least 11 four-points. The balance of our discussion will be concerned primarily with how these 11 four-points are distributed among the remaining five lines on the point 39. For this purpose, the layout shown in Figure 1 is particularly helpful, because, clearly, the third and fourth vertices of any four-point having 1 and 38 as diagonal points and l_8 as one of its sides are symmetrically located with respect to l_8. Carrying these observations a step further, we have the following almost obvious lemmas:

LEMMA 7. *A line on the point* 39, *distinct from* l_1 *and* l_8, *contains either* 0, 1, *or* 3 *pairs of points symmetrically located with respect to* l_8.

LEMMA 8. *There cannot be two lines on the point* 39 *such that each contains exactly one pair of points symmetrically located with respect to* l_8 *and such that the remaining points on either line are the reflections in* l_8 *of the remaining points on the other line.*

LEMMA 9. *There cannot be two lines on the point* 39 *such that one contains no points symmetrically located with respect to* l_8 *and one contains exactly one pair of points symmetrically located with respect to* l_8 *and such that the remaining points on the two lines constitute a set of points which is symmetric with respect to* l_8.

Suppose now that the number of four-points having 1, 38, 39 as diagonal points and l_8 as one side (which must be at least 11) is exactly 11. Then by Lemma 7, the remaining sides of these four-points must be distributed among the remaining five lines on 39 according to the scheme (3, 3, 3, 1, 1). Moreover, the points on the three lines which serve as the sides of three of the four-points occur in pairs which are symmetrically located with respect to l_8. Hence the remaining 8 points, which must be distributed among the two sides which belong to exactly one of the four-points, constitute a set which is its own reflection in l_8. However, by Lemma 8, this is impossible. Hence we conclude that no line, such as l_8, can be a side of exactly 11 of the four-points under consideration.

Suppose next that l_8 is a side of more than 11 of the four-points. The remaining sides of the four-points must then be distributed according to the scheme $(3, 3, 3, 3, x)$ where x is either 0, 1, 2 (immediately ruled out by Lemma 7), or 3. Each of these except (3, 3, 3, 3, 3) is clearly impossible, however. In fact, the points on the four lines which are sides of three four-points all occur in pairs symmetric with respect to l_8. Hence the remaining six points, which must all lie on the last line, are also three such symmetric pairs, and the pattern must therefore be (3, 3, 3, 3, 3).

Finally, let us suppose that l_8 is a side of 15 of the four-points, that is, let us suppose that l_8 contains 30 vertices. Since there are at least 124 vertices to be distributed among the six lines distinct from l_1 on the point 39, and since, as we have just seen, no line can contain exactly 22, 24, 26, or 28 vertices, and since by Lemma 9 no line can contain exactly 20 vertices, it follows that there is at least one other line on the point 39 which contains 30 vertices. If l is such a line, we may without loss of generality suppose that l contains the points 6, 13, 16, 23, 26, 33, 39.

Since l is a side of 15 of the four-points, it follows that every line on 39 except l_1 and l must be a side of exactly three four-points having l as one side. Let $l' \neq l_8$ be such a line, and let us suppose, for definiteness, that l' is a side of the four-point determined by the vertices 6 and 16 on l. Then l' must contain the opposite vertices of this four-point, namely 4 and 18. Moreover, from the relation of l' to the four-points having l_8 as one side, it is clear that the points on l' occur in pairs which are symmetrically located with respect to l_8. Hence l' must also contain the points 11 and 25. Thus the requirement that l' be a side of one of the four-points having l as a side has forced l' to contain not two but four points. Similarly, another pair of points on l which are vertices of a four-point having l' as a side will force l' to contain four more points, each of which, perforce, is distinct from 4, 11, 18, 25, unless this second pair happens to be 13, 23. In this case, however, any third pair of points on l which are vertices of a four-point having l' as a side will force l' to contain four additional points. Since l' contains only 7 points, we have thus reached a final contradiction which proves that PG_7 cannot exist.

Unfortunately, because of the proliferation of special cases, the authors have been unsuccessful in extending the methods of this paper to the investigation of the existence or nonexistence of PG_{11}. However, implemented by a computer, they might prove useful.

This paper was written while the authors were affiliated with the University of Utah. Mr. Heath is now at Weber State College, Ogden, Utah.

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ON THE SEPARABILITY OF THE RICCATI DIFFERENTIAL EQUATION

HARRY SILLER, Hofstra University

Several papers have recently appeared dealing with sufficient conditions that certain transformations of variables in the general Riccati equation

$$y' = f(x) + g(x)y + h(x)y^2 \tag{1}$$

yield a differential equation in which the variables are separable. The intent of this paper is to show that all these results, as well as some others, may be obtained more simply by first considering the so-called reduced Riccati equation; and second, by relating to the problem of separability an older result which Mitrinovitch had obtained for another purpose.

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We begin with a brief summary of the results hitherto obtained. Rao [1] finds that the transformation $y = uv - g/h$, where v is chosen so as to satisfy $v' +$

$gv=0$ will give rise to a separable equation provided that

$$\frac{hW' - (3h' - 2gh)W}{2h^{1/2}W^{1/2}} = \text{constant} \tag{2}$$

where $W=fh^2+g'h-gh'$.

Allen and Stein [2] show that the transformation $y=u\sqrt{f/h}$ will serve to separate the variables provided that

$$\frac{g + h'/2h + f'/2f}{\sqrt{fh}} = \text{constant.} \tag{3}$$

Wong [3], by considering the general substitution $y=uv-w$ and the consequent conditions on v and w,

$$\begin{aligned} w' - gw + hw^2 + f &= \alpha hv^2 \\ v' - gv + 2whv &= -\beta hv^2 \end{aligned} \tag{4}$$

where α and β are constants, obtains the results in [1] and [2] by selecting particular solutions of (4). It might be noted in passing that the system (4) is not very useful for determining v and w, since each of the equations is itself a Riccati equation even more complicated than (1).

In a generalization of the results in [1], Rao and Ukidave [4] prove that the transformation $y=uv-g/h$ yields a separable equation provided that constants a and b, and a function $v(x)$, can be found such that

$$\begin{aligned} \phi &= ahv^2 \\ v' + gv &= b\phi, \end{aligned} \tag{5}$$

where $\phi=f+(g/h)'$.

Now, it is well known [5] that (1) may be transformed into

$$Y' = F + HY^2 \tag{6}$$

by means of the change of variable $y=GY$, where $G=\exp(\int g dx)$. Equation (6) will hereafter be referred to as the "reduced Riccati equation" (RRE) corresponding to (1). The coefficients in the RRE are obviously related to those in (1) by

$$F = \frac{f}{G}, \qquad H = Gh. \tag{7}$$

It now turns out that the various results referred to above are, with no loss in generality, more simply obtained by using the RRE rather than the given equation (1).

THEOREM 1. (Analogue of [4].) *If there exist constants* λ *and* μ *and a function* $v(x)$ *such that*

$$v' = \lambda F, \qquad \mu Hv^2 = F, \tag{8}$$

then the transformation $Y=vu$ *yields an equation in which the variables are separable.*

Proof. Direct substitution from (8) into (6) gives

$$vu' - Hv^2u^2 = F - v'u. \tag{9}$$

The variables u and x in the right member can now be separated if we set $v' = \lambda F$ where λ is some constant. Then (9) becomes

$$vu' = Hv^2u^2 + F(1 - \lambda u). \tag{10}$$

Now, again, if we set $\mu Hv^2 = F$, where μ is some constant, we get

$$u' = Hv[u^2 + \mu(1 - \lambda)] \tag{11}$$

and in this equation the variables u and x are separable once v is replaced by $\sqrt{F/\mu H}$.

To illustrate further the simplicity which the use of the RRE affords, as well as to provide us with both an example and a counterexample further on (see Theorem 4), we avail ourselves of two Riccati equations from the references already cited.

Allen and Stein [2] consider the equation

$$\begin{gathered} y' = \exp(x^{4/3}) + \left(1 + \frac{4x^{1/3}}{3}\right)y + \exp(-x^{4/3})y^2 \\ y(0) = -1/2, \end{gathered} \tag{12}$$

which satisfies the criterion (3) with the constant equal to 1. For the reduction to the RRE, we readily find $G = \exp(x + x^{4/3})$, $F = e^{-x}$, $H = e^x$. The RRE (6) is

$$Y' = e^{-x} + e^x Y^2, \qquad Y(0) = -1/2. \tag{13}$$

To determine v, we have $v' = \lambda e^{-x}$ and $\mu e^x v^2 = e^{-x}$. Taking $\lambda = -1$, $\mu = 1$, we obtain $v(x) = e^{-x}$ so that (13) becomes

$$u' = u^2 + 2u, \qquad u(0) = -1/2, \tag{14}$$

and this equation is separable.

Our second illustration is provided by the Riccati equation

$$y' = xe^{-2x} - y + xy^2, \qquad y(0) = 0, \tag{15}$$

suggested by Wong [3], who points out that this equation, unlike (12), does not comply with (3). Here, too, the RRE leads rapidly to the separation of variables. We have $G = e^{-x}$, $F = xe^{-x}$, $H = xe^{-x}$ and the RRE (6) is

$$Y' = xe^{-x}(1 + Y^2), \qquad Y(0) = 0, \tag{16}$$

which is a separable equation without further ado. This example is particularly interesting since we do not even need Theorem 1. The reduction to the RRE is sufficient. In fact, it can be readily verified that the function $v(x)$ required by the theorem does not exist for (15). The reason for this will appear as a consequence of Theorem 4 below.

THEOREM 2. *The function*

$$p = \frac{g + \frac{1}{2}\left(\frac{h'}{h} - \frac{f'}{f}\right)}{\sqrt{fh}}$$

is invariant when (1) *is reduced to its* RRE.

Proof. From (7) we find

$$\frac{f'}{f} = \frac{G'}{G} + \frac{F'}{F}, \qquad \frac{h'}{h} = -\frac{G'}{G} + \frac{H'}{H}, \qquad fh = FH.$$

If we take note of the fact that $G'/G = g$, and if we write

$$P = \frac{\frac{1}{2}\left(\frac{H'}{H} - \frac{F'}{F}\right)}{\sqrt{FH}}$$

we find that $p = P$, as stated in the theorem. (In the RRE, the coefficient corresponding to g is, of course, zero.)

THEOREM 3. *A necessary and sufficient condition that*

$$\frac{H'}{H} - \frac{F'}{F} = k\sqrt{FH}, \qquad k \text{ constant}, \tag{17}$$

is that

$$F = \frac{mH}{J^2}, \qquad m \text{ constant}, \tag{18}$$

where $J = \int H dx$.

Proof. (17) may be written as $(H/F)' = kH(H/F)^{1/2}$, and, by integration, is easily seen to imply (18). Conversely (17) may be recovered from (18). It is important to note that (17) says that the invariant P is constant.

THEOREM 4. *If* λ, μ, v *of* (8) *exist, then* F *and* H *satisfy* (17).

Proof. Elimination of v between the two equations of (8) gives

$$\frac{H'}{H} - \frac{F'}{F} = -2\lambda\sqrt{\mu}\sqrt{FH}. \tag{19}$$

This brings us back to the reason why $v(x)$ fails to exist for the Riccati equation (12). For if it did exist for that equation, then (17) would have to hold; but we have already seen that it does not.

We observe, by the way, that (17) implies that F and H must have the same sign in the interval for which the solution to (6) is to be valid. Since $\sqrt{fh}$ enters into p exactly as $\sqrt{FH}$ enters into P and since G is always positive, (7) implies that the same remark applies to f and h in (1). We observe, too, that (19) implies that the constant μ in (8) is positive. (If μ is zero, then so is F, and (6) is immediately separable.)

In a paper [6] published thirty years ago, Mitrinovitch discusses the number of quadratures required to integrate (1). It turns out that one of his results can be brought to bear on the question of separability as well. The following is the relevant theorem, wherein we again use the RRE (6) instead of (1).

THEOREM 5. (Mitrinovitch) *If functions θ and ϕ can be found such that*

$$H = \frac{\theta}{2\phi} \tag{20}$$

$$F = H\phi^2 - \theta\phi + \phi \tag{21}$$

are both satisfied, then *$Y=\phi=\theta/2H$ is a solution of* (6).

Proof. Elimination of θ between (20) and (21) gives $\phi' = F + H\phi^2$, which is (6) with Y replaced by ϕ.

THEOREM 6. *If, in Theorem* 5, *$\theta = m\sqrt{FH}$, where m is a constant, then the invariant P is constant.*

Proof. From the hypothesis it follows that

$$\phi = \frac{m\sqrt{FH}}{2H} = \frac{m}{2}\sqrt{\frac{F}{H}},$$

and a simple computation shows that (20) and (21) then imply that

$$\frac{1}{\sqrt{FH}}\left(\frac{H'}{H} - \frac{F'}{F}\right) = -\frac{4+m^2}{m}, \tag{22}$$

which is the same as (17). Thus, P is constant.

THEOREM 7. *In the reduced Riccati equation* (6), *the variables can be made separable by the change of variable $Y=\phi u$, if* (20) *and* (21) *are satisfied.*

Proof. If, in (6), we write $Y=\phi u$ while taking account of (20) and (21), we find after some reduction that

$$u' = (u-1)\left[H\phi(u+1) - \frac{\phi'}{\phi}\right]. \tag{23}$$

The variables will be separable in (23) if

$$H\phi = \frac{k\phi'}{\phi}, \qquad k \text{ constant}, \tag{24}$$

$$\phi = \frac{1}{cJ}, \qquad c \text{ constant}, \tag{25}$$

where, as before, $J=\int H dx$. With this choice of ϕ we obtain

$$F = -\frac{2H}{cJ^2}$$

which is the same as (18).

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ON CONVEX POLYHEDRA

D. R. CHAND and S. S. KAPUR, Lockheed—Georgia Company

Introduction. The problem of determining the convex hull of a set of points lying in an n-dimensional Euclidean space, E_n, arises frequently in engineering applications. In the aerospace industry, the strength envelope of a set of load cases measured at specific stations of the aircraft is generated by computing the convex hull of a reasonably large number of three dimensional points. The need for a numerically efficient algorithm to generate the strength envelope prompted the authors of this paper to investigate the n-dimensional enveloping problem and to develop a systematic procedure that can be programmed on a digital computer.

The material presented here is restricted to the case of three dimensional points. The basic idea of our approach can be formally extended to treat the general n-dimensional problem as long as each face of the polytope is a simplex. For $n>3$, the degenerate case requires special attention and makes the general problem quite complex.

A straightforward approach for computing the convex polyhedron of a given set S of three dimensional points is to form all possible planes, defined by choosing every combination of three points of S, and then to test whether or not these planes bound the set S. For a typical problem that one encounters in a practical case, this straightforward approach becomes infeasible due to excessive computer run time. The method proposed in this paper is based on the observation that exactly two faces of the desired convex polyhedron $C(S)$ of a set S intersect along each edge of $C(S)$. Therefore, if one edge and one of the two faces containing this edge are known, then a rotation of the known face about the given edge through an appropriate angle determines the second face. We say an edge

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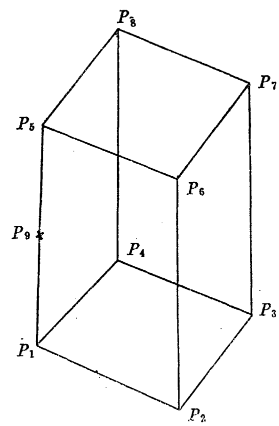

FIG. 1.

The two dimensional convex hull of all points of S on the face $P_1P_2P_3P_4$ determines four open edges, namely P_1P_2, P_2P_3, P_3P_4 and P_4P_1. The second step is to choose one of these open edges, say P_1P_4, and bend the paper around this edge P_1P_4 to cover the adjacent face $P_1P_4P_8P_5$. This operation is identical to that of rotating the face $P_1P_2P_3P_4$ about the edge P_1P_4 until the adjacent face is located. The appropriate angle of rotation will be provided by our theory. The determination of the new face $P_1P_4P_8P_5$ generates edges of the desired polyhedron that may or may not be open. In our case the edge P_1P_4 is closed and will be omitted from further consideration. This process of rotating the face or bending the paper around an open edge is continued until every edge is closed. It is quite easy to see that this procedure does generate the desired convex polyhedron of S. From this illustration it readily follows that for the procedure to succeed, our theory must provide a way: (a) to generate a starting face, (b) to compute the appropriate angle of rotation.

Definitions. We begin by formulating a few definitions that are needed to prove the main theorem of this paper. Let S be a set of n distinct points P_i (x_i, y_i, z_i) lying in E_3.

DEFINITION 1. *The convex polyhedron $C(S)$ of the set $S \subset E_n$ is the intersection of all closed convex sets in E_3 containing S.*

DEFINITION 2. *Let $\hat{u}_i$ denote a unit vector along $\overrightarrow{QP_i}$, for each $P_i \in S$, Q being a point on a plane H. We say H bounds the set S if and only if either the inner product $(\hat{n}\cdot\hat{u}_i) \geqq 0$ holds for each $P_i \in S$, or $(\hat{n}\cdot\hat{u}_i) \leqq 0$ holds, where $\hat{n}$ is a unit vector normal to H.*

DEFINITION 3. *A plane H is called a support plane of S if H bounds S and at least one point of S lies on H.*

DEFINITION 4. *A plane H is called a face of the convex polyhedron $C(S)$ if H is a support plane of S containing at least three independent points of S.*

DEFINITION 5. *The line segment P_iP_j, $P_i, P_j \in S$, is an edge of the convex polytope $C(S)$ if there exists a support plane of S, containing P_iP_j, which is not a face of $C(S)$.*

MAIN THEOREM. *There are exactly two faces of $C(S)$ which intersect along the edge P_IP_J of $C(S)$.*

Proof. We shall give a constructive proof of this theorem. P_IP_J being an edge of $C(S)$ implies the existence of a support plane H of S, containing P_IP_J, which is not a face of $C(S)$. There is no loss of generality in assuming that the unit normal $\hat{n}$ to H satisfies the condition

$$(1) \qquad (\hat{n}\cdot\hat{v}_k) > 0, \qquad \begin{array}{l} k = 1, 2, \cdots, n \\ k \neq I, J \end{array}$$

where $\hat{v}_k$ is a unit vector along $\overrightarrow{P_IP_k}$, for all $k \neq I$. Construct a unit vector $\hat{e}$ as follows:

$$(2) \qquad \hat{e} = \hat{n} \times \hat{v}_J.$$

Let $\hat{n}_k$ be a unit vector normal to the plane spanned by adjoining to P_I, P_J, the point $P_k \in S$ so that P_I, P_J, P_k form an independent set. The direction of $\hat{n}_k$ is

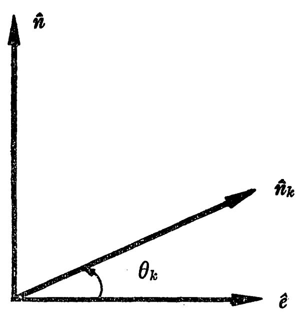

FIG. 2.

specified by the relation

$$(\hat{e}\cdot\hat{n}_k) > 0. \tag{3}$$

The vector $\hat{n}_k$, being orthogonal to v_J, must lie in the plane spanned by $\hat{n}$ and $\hat{e}$, as shown in Figure 2. Therefore, we can write

$$\hat{n}_k = \lambda_k\hat{n} + \mu_k\hat{e} \tag{4}$$

where,

$$1 = \lambda_k^2 + \mu_k^2 \tag{5}$$

and

$$\theta_k = \text{Arc}\tan(\lambda_k/\mu_k). \tag{6}$$

Since $\hat{n}_k$ is orthogonal $\hat{v}_k$, $(\hat{n}_k\cdot\hat{v}_k)=0$. That is, $\lambda_k(\hat{n}\cdot\hat{v}_k)+\mu_k(\hat{e}\cdot\hat{v}_k)=0$.

Using relations (1) and (3), and the fact that P_I, P_J, P_k form a linearly independent set, we can write

$$\frac{\lambda_k}{\mu_k} = -\frac{(\hat{e}\cdot\hat{v}_k)}{(\hat{n}\cdot\hat{v}_k)} \tag{7}$$

for each appropriate $P_k\in S$.

Suppose the plane F_m spanned by P_I, P_J and $P_m\in S$ is a face of $C(S)$. Then there are exactly two possibilities, either $(\hat{n}_m.\hat{v}_k)\geqq 0$ or $(\hat{n}_m\cdot\hat{v}_k)\leqq 0$, for each $P_k\in S$. But

$$(\hat{n}_m\cdot\hat{v}_k) = \lambda_m(\hat{n}\cdot\hat{v}_k) + \mu_m(\hat{e}\cdot\hat{v}_k). \tag{8}$$

Using relations (1) and (7), and the fact $\mu_m\neq 0$, we can rewrite (8) as follows:

$$(\hat{n}_m\cdot\hat{v}_k) = \lambda_m(\hat{n}\cdot\hat{v}_k)\left[\frac{\lambda_m}{\mu_m} - \frac{\lambda_k}{\mu_k}\right]$$

for all $P_k\epsilon S$ and $k\neq I, J$.

Since $\mu_m>0$ and $(\hat{n}\cdot\hat{v}_k)>0$, it follows that either $\lambda_m/\mu_m\geqq\lambda_k/\mu_k$ or $\lambda_m/\mu_m\leqq\lambda_k/\mu_k$, for each $P_k\in S$. Let P_r and P_t be two points of S such that

$$\frac{\lambda_r}{\mu_r} = \min_{\substack{1\leqq k\leqq n\\ k\neq I,J}}.\ \{\lambda_k/\mu_k\} \tag{9}$$

and

$$\frac{\lambda_t}{\mu_t} = \max_{\substack{1\leqq k\leqq n\\ k\neq I,J}}.\ \{\lambda_k/\mu_k\}. \tag{10}$$

Then F_r and F_t are the desired two faces of $C(S)$ which intersect along the known edge P_IP_J. From our construction it is evident that there are exactly two such faces of $C(S)$.

Refinements. *Remark* (1). Suppose a face F_r containing the edge P_IP_J of

$C(S)$ is given. Then the face F_t adjacent to F_r along P_IP_J is generated as follows:

Let $\hat{n}$ denote the unit normal to the known face F_r such that $(\hat{n}\cdot\hat{v}_k)\geqq 0$, $k=1, 2, \ldots, n$. Define the unit vector $\hat{e}=\hat{n}\times\hat{v}_J$ such that $(\hat{e}\cdot\hat{v}_k)\geqq 0$, for all $P_k\in S$. Compute the ratio

$$\frac{\lambda_k}{\mu_k} = -\frac{(\hat{e}\cdot\hat{v}_k)}{(\hat{n}\cdot\hat{v}_k)}, \qquad k = 1, 2, \cdots, n.$$

Let $P_t\in S$ be a point such that

$$\frac{\lambda_t}{\mu_t} = \max_{1\leqq k\leqq n}\left\{\frac{\lambda_k}{\mu_k}\right\}.$$

Then the plane F_t spanned by P_I, P_J and P_t is the desired face of $C(S)$ whose normal $\hat{n}_t$ is given by

$$\hat{n}_t = \lambda_t\hat{n} + \mu_t\hat{e}.$$

Note the second face corresponding to the minimum value of the ratio λ_k/μ_k is the face F_r itself, since for a point $P_r\in S$ on F_r, $(\hat{n}\cdot\hat{v}_r)=0$, and the ratio tends to minus infinity.

Remark (2). The construction of the second face in Remark (1) provides the answer to question (b) raised above. The angle ϕ_k between the known face F_r and plane F_k, which is a candidate for being a face of $C(S)$, is given by the relation

$$\phi_k = \pi/2 + \theta_k$$

where θ_k is defined in equation (6). Since $0 \leqq\phi_k\leqq\pi$, it follows $-\pi/2\leqq\theta_k\leqq\pi/2$. Therefore, maximizing the ratio λ_k/μ_k is equivalent to maximizing $\tan\theta_k$, $-\pi/2\leqq\theta_k\leqq\pi/2$, which in turn implies maximizing θ_k and hence maximizing ϕ_k.

Remark (3). One can determine the angle ϕ_k by computing the angle between the normals to the known face F_r and the plane F_k. To determine the face F_t one needs to generate the normals to each of the plane F_k, which is time consuming, whereas in our approach we construct exactly one new vector $\hat{e}$ to generate a new face. This cuts the execution time of the computer program immensely.

Remark (4). It was mentioned that each face automatically produces at least two edges of $C(S)$ that are different from the edge of rotation. This fact can be easily verified as follows: The face F_t in Remark (1) is computed by determining a point $P_t\in S$ for which the ratio of λ_k/μ_k assumes its maximum value. If P_t is unique, then we assert the line segments P_IP_t and P_JP_t are edges of $C(S)$. To prove P_IP_t is an edge, we need to show the existence of a support plane of S containing P_IP_t which is not a face.

Since the line segment P_IP_t bounds the points of S which are on the face F_t, this face F_t can be rotated about P_IP_t through a sufficiently small angle so that the plane F_t in its new position still bounds S and no point of S, except P_I and P_J, lies on it. Hence, P_IP_t is an edge of $C(S)$. In the case when P_t is not unique, determine the two dimensional convex hull of these planar points P_I, P_J and P_t's by either brute force or by our algorithm adapted for two dimensions. Then by a similar reasoning, it can be verified that each side of the resulting polygon is an edge of $C(S)$.

Starting Face. The following procedure may be adopted for generating a starting face. Determine the lowest point(s) of the set S. Let $P_I \in S$ be a point such that

$$z_I = \min_{1 \leq k \leq n} \{z_k\}. \tag{11}$$

If there exists another point $P_J \in S$ for which

$$z_J = z_I$$

then the line segment $P_I P_J$ becomes an edge of $C(S)$ and the plane $z = z_I$ is a support plane of S containing $P_I P_J$. Thus an application of our main theorem generates a face. However, if there are more than two independent points with same minimum z value, they define a face of $C(S)$. Suppose P_I is unique. Then compute $P_J \in S$ as follows:

$$Q_J = \min_{\substack{1 \leq k \leq n \\ k \neq I}} \left\{ \frac{y_k - y_I}{z_k - z_I} \right\}. \tag{12}$$

If the minimum in (12) is assumed at a unique point P_J, then we assert $P_I P_J$ is an edge of $C(S)$. Let $\vec{n}$ be the normal to the plane H spanned by the vectors $P_I P_J$ and $\hat{\imath}$.

$$\vec{n} = -(z_J - z_I)\hat{\jmath} + (y_J - y_I)\hat{k}.$$

For any point $P_m \in S$, $m \neq I, J$

$$\begin{aligned}(\vec{n} \cdot \overrightarrow{P_m P_I}) &= -(z_J - z_I)(y_m - y_I) + (y_J - y_I)(z_m - z_I) \\ &= (z_J - z_I)(z_m - z_I)(Q_J - Q_m) \\ &< 0\end{aligned}$$

using (11) and (12). This shows that H is a support plane of S, containing $P_I P_J$, which is not a face of $C(S)$. An application of the main theorem determines a face of $C(S)$. On the other hand if P_J is nonunique, the two dimensional convex hull generates an edge which in turn is used to generate a face of $C(S)$.

Numerical example. Let S be the set of points P_i defined as follows:

i	1	2	3	4	5	6	7	8	9	10
x_i	0	0	-1	-1	0	0	-1	-1	0	$-\frac{1}{2}$
y_i	0	1	1	0	0	1	1	0	0	$\frac{1}{2}$
z_i	0	0	0	0	1	1	1	1	$\frac{1}{2}$	$\frac{1}{2}$

to compute the convex polyhedron of S. A plot of the points of S will show that the desired convex polyhedron is the box of Figure 1.

Step 1. Generation of a starting face. There are four points P_1, P_2, P_3, P_4 with minimum z-component. These four points define the plane $z = 0$ which is the starting face. The normal $\hat{n}$ to the starting face is of the form

$$\hat{n} = (0, 0, 1) = \hat{k}.$$

The convex hull of the projection of P_1, P_2, P_3, P_4 on the $z=0$ plane shows that P_1P_2, P_2P_3, P_3P_4 and P_4P_1 are four open edges.

Step 2. Choose the edge P_1P_2. A unit vector along $\overrightarrow{P_1P_2}$ is given by

$$\hat{v}_2 = (0, 1, 0) = \hat{\jmath}.$$

Construct the $\hat{e}$ vector of the main theorem.

$$\hat{e} = \hat{n} \times \hat{v}_2 = (-1, 0, 0) = -\hat{\imath}.$$

Verify that $\hat{e}$ has the direction specified in Remark (1). That is, $(\hat{e}\cdot\hat{v}_k) \geqq 0$, $k = 1, 2, \ldots, 10$, $\hat{v}_k$ being a unit vector along $\overrightarrow{P_kP_1}$.

Step 3. Using the vectors $\hat{n}$ and $\hat{e}$, compute the ratio $R_k = \lambda_k/\mu_k$ for $k=3, 4, \ldots, 10$.

$$R_k = -\frac{(\hat{e}\cdot\hat{v}_k)}{(\hat{n}\cdot\hat{v}_k)}.$$

R_3, R_4 tend to minus infinity; $R_5 = R_6 = R_9 = 0$; $R_7 = R_8 = R_{10} = -1$.

Thus the ratio assumes its maximum value at three points P_5, P_6 and P_9. The convex hull of P_1, P_2, P_6, P_6 and P_9 in the two dimensional case shows that P_1P_2, P_2P_5, P_5P_6, P_6P_1 are the four edges. The degeneracy, caused by the fact that P_9 lies on an edge, can be taken care of in the two dimensional case simply by comparing distances. Since we found the two faces containing the edge P_1P_2, we say P_1P_2 is closed and we eliminate it from future consideration. This completes one cycle.

Step 4. Check whether there is an open edge among the edges generated in the last cycle. In our case we have three open edges, namely, P_2P_5, P_5P_6, P_6P_1. Pick any one of these edges and after storing the remaining open edges for future consideration, return to Step 2 with the normal $\hat{n} = \hat{n}^*$, where

$$\hat{n}^* = \lambda_m\hat{n} + \mu_m\hat{e}.$$

In our example, λ_m and μ_m in the last cycle correspond to λ_5 and μ_5.

On the other hand, if there is no open edge among the edges generated in the last cycle, search the storage for an open edge. Pick an open edge, if it exists, and after computing the normal to the face containing this edge, return to Step 2. No open edge in storage implies the desired convex polyhedron has been generated. The results of each cycle are tabulated below.

Cycle	Rotation About Edge	Face Generated	Edges of the Face Generated	Edges in Storage after the Cycle
0	x	$P_1P_2P_3P_4$	P_1P_2, P_2P_3, P_3P_4, P_4P_1	P_2P_3, P_3P_4, P_4P_1
1	P_1P_2	$P_1P_2P_6P_5P_9$	P_1P_2, P_2P_6, P_6P_5, P_5P_1	P_2P_3, P_3P_4, P_4P_1, P_6P_5, P_5P_1
2	P_2P_6	$P_2P_6P_7P_3$	P_2P_6, P_6P_7, P_7P_3, P_3P_2	P_3P_4, P_4P_1, P_6P_5, P_5P_1, P_7P_3
3	P_6P_7	$P_6P_7P_8P_5$	P_6P_7, P_7P_8, P_8P_5, P_5P_6	P_3P_4, P_4P_1, P_5P_1, P_7P_3, P_8P_5
4	P_7P_8	$P_7P_8P_4P_3$	P_7P_8, P_8P_4, P_4P_3, P_3P_7	P_4P_1, P_5P_1, P_8P_5
5	P_8P_4	$P_8P_4P_1P_9P_5$	P_8P_4, P_4P_1, P_1P_5, P_5P_8	None

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MORLEY'S TRIANGLE THEOREM

R. J. WEBSTER, University of Sheffield

Let ABC be a triangle with inradius r and circumradius R, and let the adjacent trisectors of angles A, B, C meet in A', B', C' as illustrated in Figure 1. Then Morley's theorem states that the triangle A', B', C' is equilateral. The proofs of this theorem, which are usually given, do not include a calculation of

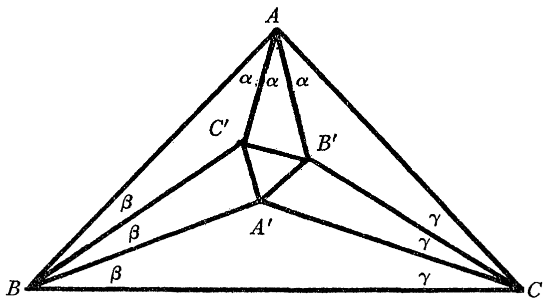

FIG. 1.

the side of the equilateral triangle $A'B'C'$. In this note we prove Morley's theorem by showing that the side of this triangle is $8R \sin A/3 \sin B/3 \sin C/3$. This is a particularly interesting result, when one recalls the formula $r=4R \sin A/2 \sin B/2 \sin C/2$.

In any triangle ABC, we have $a=2R \sin A$, etc. Let $A=3\alpha$, $B=3\beta$, $C=3\gamma$. Then applying the sine rule to $A'BC$ we obtain

$$\begin{aligned} A'B &= 2R\sin 3\alpha \sin\gamma/\sin(\beta+\gamma),\\ &= 2R(4\sin\alpha\sin(60+\alpha)\sin(60-\alpha)\sin\gamma/\sin(60-\alpha),\\ &= 8R\sin\alpha\sin\gamma\sin(60+\alpha). \end{aligned}$$

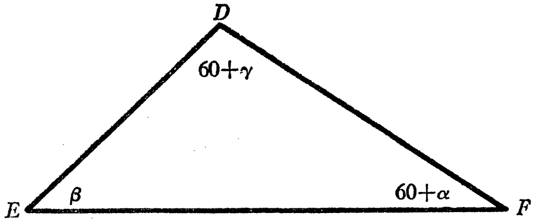

FIG. 2.

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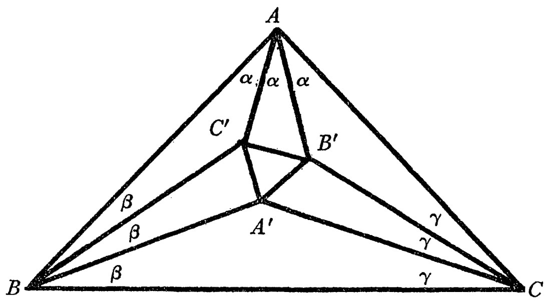

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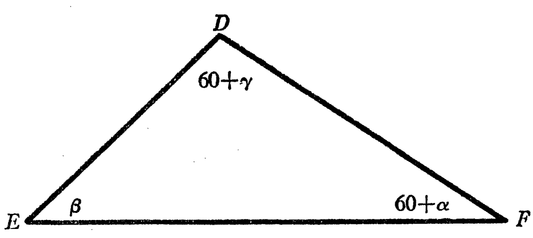

FIG. 2.

Similarly $BC'=8R \sin\alpha \sin\gamma \sin(60+\gamma)$. Consider now the triangle DEF shown in Figure 2, where $DE=8R \sin\alpha \sin\gamma \sin(60+\alpha)$. It follows, using the sine rule on DEF, that $EF=8R \sin\alpha \sin\gamma \sin(60+\gamma)$ and $DF=8R \sin\alpha \sin\beta \sin\gamma$. However, the triangles $A'BC'$ and DEF are congruent side-angle-side, so $A'C'=DF=8R \sin\alpha \sin\beta \sin\gamma$. Thus, by the symmetry of this expression, the triangle $A'B'C'$ is equilateral and Morley's theorem is proved.

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MORLEY'S TRIANGLE

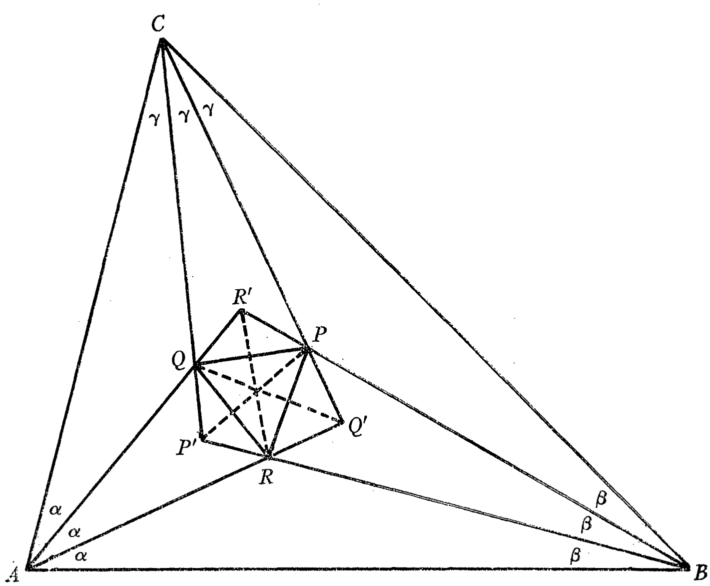
J. C. BURNS, Australian National University

The trigonometrical approach to Morley's theorem used in [1] can be extended to provide more information about the configuration and an alternative proof of the theorem itself.

The diagram and the notation are the same as in [1] except that the following additional points of intersection of the trisectors of the angles are defined:

$$P' = (BR, CQ), \qquad Q' = (CP, AR), \qquad R' = (AQ, BP).$$

It will be shown that triangles $P'QR$, $Q'RP$, $R'PQ$ are all isosceles and hence



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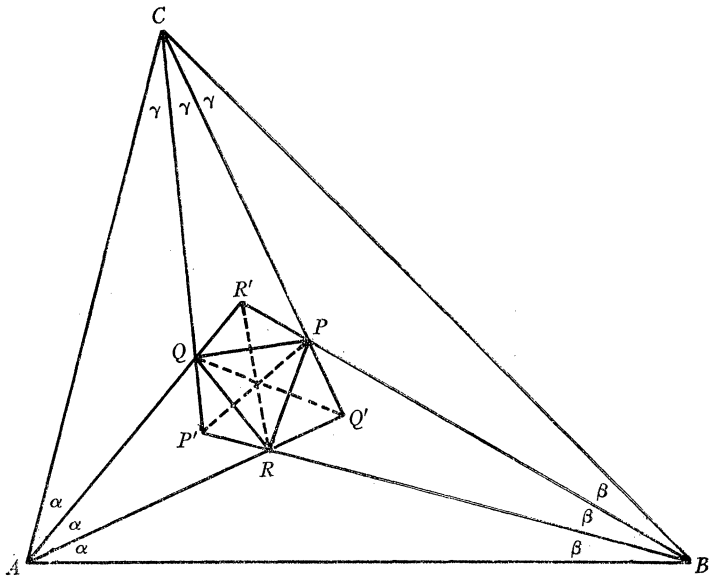

that PP', QQ', RR' are the perpendicular bisectors of the sides of the Morley triangle and so are concurrent and equally inclined to one another.

From (8) of [1],

$$QC = 4D \sin \alpha \sin \beta \sin (60° + \beta)$$

where D is the diameter of the circumcircle of triangle ABC.

The sine rule applied in triangle $P'BC$ and identity (1) of [1] can be used to show that $P'C = 4D \sin \alpha \sin \beta \cos \beta \sin (60°+\alpha)/\cos(60°-\alpha)$. $P'Q = P'C - QC$ is then obtained, after some simplification, as

$$P'Q = 2D \sin \alpha \sin \beta \sin \gamma / \cos (60° - \alpha).$$

Since this is symmetrical in β and γ, it follows that $P'Q = P'R$ and triangle $P'QR$ is isosceles. In the same way, it can be proved that triangles $Q'RP$ and $R'PQ$ are isosceles.

From the triangle $P'BC$ it is seen that angle $QP'R = 180° - 2(\beta+\gamma)$ so that each of the base angles of the isosceles triangle $P'QR$ is equal to $\beta+\gamma = 60° - \alpha$. Hence

$$QR = 2P'Q \cos (60° - \alpha) = 4D \sin \alpha \sin \beta \sin \gamma$$

(as in the last sentence of [1]). The symmetry of this expression for QR shows that the triangle PQR is equilateral so that an alternative proof of Morley's theorem is obtained.

Because the triangle $P'QR$ is isosceles and the triangle PQR is equilateral, the perpendicular bisector of QR passes through both P and P'. Similar results hold for the other sides of triangle PQR so that PP', QQ', RR', being the perpendicular bisectors of the sides of triangle PQR, are concurrent and equally inclined to one another.

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AXIS ROTATION VIA PARTIAL DERIVATIVES

PETER HAGIS, JR., Temple University

The following derivation of the familiar rotation of axes formulae in two dimensions may prove to be of interest as an application of some of the concepts encountered when one begins the study of functions of several variables.

Thus, suppose that two rectangular coordinate systems, an $X-Y$ system and a $U-V$ system, have the same origin and that the angle XOU is α. If (x, y) and (u, v) denote the coordinates of the same point with respect to the two systems then it is geometrically obvious that both x and y are continuous functions of the two variables u and v. We shall determine these functions explicitly by first finding their partial derivatives.

that PP', QQ', RR' are the perpendicular bisectors of the sides of the Morley triangle and so are concurrent and equally inclined to one another.

From (8) of [1],

$$QC = 4D \sin \alpha \sin \beta \sin (60° + \beta)$$

where D is the diameter of the circumcircle of triangle ABC.

The sine rule applied in triangle $P'BC$ and identity (1) of [1] can be used to show that $P'C = 4D \sin \alpha \sin \beta \cos \beta \sin (60° + \alpha)/\cos(60° - \alpha)$. $P'Q = P'C - QC$ is then obtained, after some simplification, as

$$P'Q = 2D \sin \alpha \sin \beta \sin \gamma / \cos (60° - \alpha).$$

Since this is symmetrical in β and γ, it follows that $P'Q = P'R$ and triangle $P'QR$ is isosceles. In the same way, it can be proved that triangles $Q'RP$ and $R'PQ$ are isosceles.

From the triangle $P'BC$ it is seen that angle $QP'R = 180° - 2(\beta + \gamma)$ so that each of the base angles of the isosceles triangle $P'QR$ is equal to $\beta + \gamma = 60° - \alpha$. Hence

$$QR = 2P'Q \cos (60° - \alpha) = 4D \sin \alpha \sin \beta \sin \gamma$$

(as in the last sentence of [1]). The symmetry of this expression for QR shows that the triangle PQR is equilateral so that an alternative proof of Morley's theorem is obtained.

Because the triangle $P'QR$ is isosceles and the triangle PQR is equilateral, the perpendicular bisector of QR passes through both P and P'. Similar results hold for the other sides of triangle PQR so that PP', QQ', RR', being the perpendicular bisectors of the sides of triangle PQR, are concurrent and equally inclined to one another.

Reference

1. G. L. Neidhardt and V. Milenkovic, this MAGAZINE, 42 (1969) 87–88.

AXIS ROTATION VIA PARTIAL DERIVATIVES

PETER HAGIS, JR., Temple University

The following derivation of the familiar rotation of axes formulae in two dimensions may prove to be of interest as an application of some of the concepts encountered when one begins the study of functions of several variables.

Thus, suppose that two rectangular coordinate systems, an $X-Y$ system and a $U-V$ system, have the same origin and that the angle XOU is α. If (x, y) and (u, v) denote the coordinates of the same point with respect to the two systems then it is geometrically obvious that both x and y are continuous functions of the two variables u and v. We shall determine these functions explicitly by first finding their partial derivatives.

If we let u take on an increment Δu while holding v fixed we see that $\Delta x/\Delta u = \cos\alpha$ if Δu is positive, while $\Delta x/\Delta u = -\Delta x/|\Delta u| = -\cos(180°+\alpha) = \cos\alpha$ if Δu is negative. (The reader should draw a diagram.) Similarly, $\Delta y/\Delta u = \sin\alpha$ if Δu is positive and $\Delta y/\Delta u = -\Delta y/|\Delta u| = -\sin(180°+\alpha) = \sin\alpha$ if Δu is negative. Letting Δu approach zero we have $\partial x/\partial u = \cos\alpha$ and $\partial y/\partial u = \sin\alpha$. If we now hold u fixed and change v by Δv we see that $\Delta x/\Delta v = -\sin\alpha$ and $\Delta y/\Delta v = \cos\alpha$, whether Δv is positive or negative. (For example, if Δv is negative then $\Delta x/\Delta v = -\cos(270°+\alpha)$ and $\Delta y/\Delta v = -\sin(270°+\alpha)$.) It follows that $\partial x/\partial v = -\sin\alpha$ and $\partial y/\partial v = \cos\alpha$.

Now, since the partial derivatives of x and y with respect to u and v are continuous, both x and y are continuous and differentiable functions of u and v (see pp. 267–268 in [1]). Also, since the functions $F = u\cos\alpha - v\sin\alpha$ and $G = u\sin\alpha + v\cos\alpha$ are continuous differentiable functions of u and v and have the same partial derivatives as x and y, respectively, it follows from the Law of the Mean (see p. 280 in [1]) that $x = F + c$ and $y = G + C$ where c and C are constants. Since the two coordinate systems have a common origin we see immediately that $c = C = 0$. Therefore, the functions being sought are

$$\begin{cases} x = u\cos\alpha - v\sin\alpha, \\ y = u\sin\alpha + v\cos\alpha. \end{cases}$$

Reference

1. J. Olmsted, Advanced Calculus, Appleton-Century-Crofts, New York, 1961.

REMARK ON THE PAPER "SUMS OF SQUARES OF CONSECUTIVE ODD INTEGERS" BY BROTHER U. ALFRED

ANDRZEJ MAKOWSKI, Warsaw, Poland

In the paper [1] the proof of Theorem 7 contains an error: $(x+12m)^2 \equiv -z'^2 \pmod{6m+1}$ and $6m+1$ has a factor of the form $4k+3$ need not imply a contradiction, because both $x+12m$ and z' may be divisible by $4k+3$. In particular, the case $n = 241$ cannot be excluded by Theorem 7 (what was done in Table I) because in this case we have $m = 20$, $6m+1 = 121 = 11^2$ and the congruence $(x+240)^2 \equiv -z'^2 \pmod{121}$ has a solution $x+240 = 11t$, $z' = 11u$. Thus the case $n = 241$ should be listed as unresolved. The corrected version of Theorem 7 is as follows:

THEOREM. *If $3 \mid n-1$ and there exists a prime number p of the form $4l+3$ such that for some integer k we have $p^{2k+1} \mid n+1$ and $p^{2k+2} \nmid n+1$, then the equation*

$$3(x + n - 1)^2 + (n - 1)(n + 1) = 3nz'^2$$

has no solution in integers x, z'.

Proof. Put $x+n-1 = X$, $z' = Z$. If for some X, Z

If we let u take on an increment Δu while holding v fixed we see that $\Delta x/\Delta u = \cos\alpha$ if Δu is positive, while $\Delta x/\Delta u = -\Delta x/|\Delta u| = -\cos(180°+\alpha) = \cos\alpha$ if Δu is negative. (The reader should draw a diagram.) Similarly, $\Delta y/\Delta u = \sin\alpha$ if Δu is positive and $\Delta y/\Delta u = -\Delta y/|\Delta u| = -\sin(180°+\alpha) = \sin\alpha$ if Δu is negative. Letting Δu approach zero we have $\partial x/\partial u = \cos\alpha$ and $\partial y/\partial u = \sin\alpha$. If we now hold u fixed and change v by Δv we see that $\Delta x/\Delta v = -\sin\alpha$ and $\Delta y/\Delta v = \cos\alpha$, whether Δv is positive or negative. (For example, if Δv is negative then $\Delta x/\Delta v = -\cos(270°+\alpha)$ and $\Delta y/\Delta v = -\sin(270°+\alpha)$.) It follows that $\partial x/\partial v = -\sin\alpha$ and $\partial y/\partial v = \cos\alpha$.

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has no solution in integers x, z'.

Proof. Put $x+n-1 = X$, $z' = Z$. If for some X, Z

$$X^2 + \tfrac{1}{3}(n-1)(n+1) = nZ^2$$

or

$$X^2 - nZ^2 = -\tfrac{1}{3}(n-1)(n+1)$$

and $n \equiv -1 \pmod{p}$ then $X^2+Z^2 \equiv 0 \pmod{p}$, hence $X=pX_1$, $Z=pZ_1$ and $p^2(X_1^2-nZ_1^2)=-\frac{1}{3}(n-1)(n+1)$. We may proceed similarly and the left-hand side will be divisible exactly by p^{2u} and the right-hand side by p^{2k+1}, which is a contradiction.

We may observe that the case $n=601$ which is listed as unresolved is settled by the above theorem (we have $3 \mid 600$ and $7 \mid 602$, $7^2 \nmid 602$).

Reference

1. Brother U. Alfred, Sums of squares of consecutive odd integers, this MAGAZINE, 40 (1967) 194–199.

ON THE CONSTRUCTION OF MULTIPLE CHOICE TESTS

or

BARBAROUS PARODIES OF THE BARBER'S PARADOX

or

A MATHEMATICS INSTRUCTOR'S REALIZATION OF A MEAN TESTING METHOD

Frequently, multiple choice tests are constructed in such a manner that a scoring sheet may be used. Instructions for filling out the scoring sheet, as well as a sample question or two with illustrated answers, appear at the beginning of the test somewhat as follows:

Use the special scoring sheet and scoring pencil furnished you by the tester. For each question there are five choices for answers, labeled a, b, c, d, and e, respectively. One or more of these answers is right, the others being wrong. Use your scoring pencil, as illustrated in the following examples, to fill in solidly the space between the pair of parallel line segments which follow a letter if and only if the answer corresponding to that letter is right.

Example 1. For $x^2-x=2$,

a. There are two distinct solutions in the ring of integers.
b. There is precisely one solution in the set of natural numbers.
c. 2 is a solution.
d. 0 is a solution.
e. Precisely two of the preceding answers are right.

Example 2. For $x^2-x+1=0$,

a. There are two distinct solutions in the ring of integers.
b. There is precisely one solution in the set of natural numbers.

$$X^2 + \tfrac{1}{3}(n-1)(n+1) = nZ^2$$

or

$$X^2 - nZ^2 = -\tfrac{1}{3}(n-1)(n+1)$$

and $n \equiv -1 \pmod{p}$ then $X^2+Z^2 \equiv 0 \pmod{p}$, hence $X = pX_1$, $Z = pZ_1$ and $p^2(X_1^2 - nZ_1^2) = -\frac{1}{3}(n-1)(n+1)$. We may proceed similarly and the left-hand side will be divisible exactly by p^{2u} and the right-hand side by p^{2k+1}, which is a contradiction.

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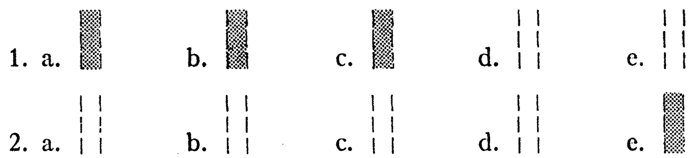
e. Precisely two of the preceding answers are right.

Example 2. For $x^2 - x + 1 = 0$,

a. There are two distinct solutions in the ring of integers.
b. There is precisely one solution in the set of natural numbers.

c. There is a solution in the field of real numbers.
d. 1 is a solution.
e. None of the preceding answers is right.

The correct solutions on the answer sheet should be:



Any other choices would be incorrect.

When constructing such a test in mathematics, the tester may unintentionally make some questions either quite difficult or relatively simple. The persons taking the test sometimes expect it to be extremely difficult. If the tester wants to fulfill such expectations he can include questions in which a slight rewording of part e will make any choice of markings an incorrect answer. This may be illustrated by use of the above two sample questions. Use the same wording as before with the exception of the last parts, which are reworded as follows:

1. e. Precisely three of these answers are right.
2. e. None of these answers is right.

As one final illustration consider the following example:

Example 3. The empty set is frequently denoted by $\varnothing$.
a. $\varnothing$ is also called the null set.
b. The answer to part a is right.
c. The answer to parts a and b are right.
d. The answer to parts a, b, and c are right.
e. Precisely four of these answers are right.

CERTAIN DISTRIBUTIONS OF UNLIKE OBJECTS INTO CELLS

MORTON ABRAMSON, York University, Toronto

Introduction. Denote by $N(n, p)$ the number of ways of distributing n unlike objects $a_1, a_2, \cdots, a_n$ into p unlike cells $c_1, c_2, \cdots, c_p$ with no cell empty. As is well known (c.f. [**4**]),

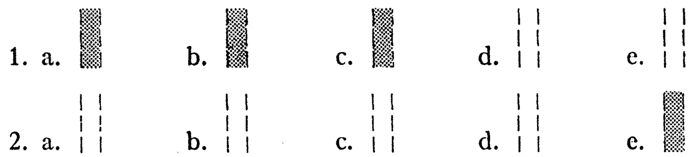
$$(1) \qquad N(n,p) = \sum_{i=0}^{p-1} (-1)^i \binom{p}{i} (p-i)^n.$$

If the cells are like cells, i.e., indistinguishable, then the number of ways of distributing the objects with no cell empty, is, of course, the well known Stirling number of the second kind,

$$(2) \qquad S(n,p) = N(n,p)/p!$$

c. There is a solution in the field of real numbers.
d. 1 is a solution.
e. None of the preceding answers is right.

The correct solutions on the answer sheet should be:



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As one final illustration consider the following example:

Example 3. The empty set is frequently denoted by $\emptyset$.
a. $\emptyset$ is also called the null set.
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$$N(n,p) = \sum_{i=0}^{p-1} (-1)^i \binom{p}{i} (p-i)^n. \tag{1}$$

If the cells are like cells, i.e., indistinguishable, then the number of ways of distributing the objects with no cell empty, is, of course, the well known Stirling number of the second kind,

$$S(n,p) = N(n,p)/p! \tag{2}$$

(see again [4]). Define the spacing of objects a_i and a_j as $|i-j|$. In [1] R. Lagrange by means of modified Vandermonde determinants finds the number $N_k(n, p)$ of distributions counted in $N(n, p)$ with the extra condition that no cell contains objects whose spacing is less than a preassigned integer k. Clearly $N_k(n, p)/p!$ is the answer when the cells are like cells or what is the same thing, the number of partitions of $\{a_1, \cdots, a_n\}$ into p mutually disjoint subsets, none empty and no subset containing objects whose spacing is less then k. Of course, $N_1(n, p)=N(n, p)$.

In this note we find the numbers $N_k(n, p)$ using a direct elementary combinatorial argument. Also, the number of distributions such that no two objects whose spacing is exactly k are in the same cell is given. A recurrence relation for $N_k(n, p)$ is given as well as other relations involving Stirling numbers of the second kind. By defining the spacing of two distinct cells c_i and c_j as $|i-j|$ certain distributions are given where any two cells, whose spacing is less than or greater than a fixed number, do not both contain objects. In the concluding remarks the analogous situation in dealing with permutations with repetition or random sampling with replacement is noted.

Spacing on objects; inequality condition. The number of ways of distributing n unlike objects $a_1, \cdots, a_n$ into r unlike cells such that a_i, a_j are not in the same cell if $|i-j| < k$ is easily seen to be

$$M_k(n,r) = r(r-1)(r-2)\cdots(r-k+2)(r-k+1)^{n-k+1}, \tag{3}$$

for, object a_1 can be put into any one of r cells. Once this is done, object a_2 can be put into any one of $r-1$ cells, $\cdots$; once this is done object a_{k-1} can be put into any one of $r-k+2$ cells, then a_{k+i} can be put in order into any one of $r-k+1$ cells, $i=0, 1, \cdots, n-k$. Now $N_k(n, p)$ is the number of permissible distributions with none of the p cells empty. For a fixed selection of i of the p cells there are $M_k(n, p-i)$ permissible distributions with that particular selection of i cells empty. Hence, using the well known principle of inclusion and exclusion we have

$$\begin{aligned} N_k(n, p) &= \sum_{i=0}^{p-k} (-1)^i \binom{p}{i} M_k(n, p-i) \\ &= \sum_{i=0}^{p-k} (-1)^i \binom{p}{i} (p-i)(p-i-1)\cdots(p-i-k+2)(p-i-k+1)^{n-k+1} \\ &= \frac{p!}{(p-k+1)!} \sum_{i=0}^{p-k} (-1)^i \binom{p-k+1}{i} (p-k+1-i)^{n-k+1} \end{aligned} \tag{4}$$

which is another form of expression (18) in [1].

From (1), (2) and (4) we have the relation

$$\begin{aligned} N_k(n,p) &= \frac{p!}{(p-k+1)!} N(n-k+1, p-k+1) \\ &= p!\,S(n-k+1, p-k+1). \end{aligned} \tag{5}$$

The recurrence relation

$$N_k(n,p) = pN_k(n-1,p-1) + (p-k+1)N_k(n-1,p) \tag{6}$$

is easily established since $pN_k(n-1,\ p-1)$ counts those distributions in which a_n is in a cell alone and $(p-k+1)N_k(n-1,\ p)$ counts those distributions in which a_n is in a cell with at least one other object, but other than one of the $k-1$ objects $a_{n-k+1}, \cdots, a_{n-1}$ no two of which are in the same cell. In fact (4) could be deduced from (6) with proper initial values.

The special case $k=1$ in (4) becomes $N_1(n,\ p)=N(n,\ p)=\sum_{i=0}^{p-1}(-1)^i\binom{p}{i}\cdot(p-i)^n$ in agreement with (1) and from (6) we have the known relation

$$N(n,\ p) = pN(n-1,\ p-1) + pN(n-1,\ p).$$

Application. The number of distributions of the n objects into the p cells with no cell containing objects whose spacing is less than k and with exactly s cells empty is, clearly,

$$N_k^s(n,p) = \binom{p}{s}N_k(n,p-s), \tag{7}$$

and from the definition of $M_k(n,\ p)$

$$\sum_{s=0}^{p-1} N_k^s(n,p) = M_k(n,p). \tag{8}$$

From (3), (4), (5), (7) and (8) we have the summation

$$\begin{aligned}&\sum_{s=0}^{p-1}\sum_{i=0}^{p-s-k}(-1)^i p!(p-s-k+1-i)^{n-k+1}/s!i!(p-s-k+1-i)!\\ &= \sum_{s=0}^{p=1} p!S(n-k+1,p-s-k+1)/s!\\ &= p(p-1)(p-2)\cdots(p-k+2)(p-k+1)^{n-k+1}.\end{aligned}$$

In the particular case $k=1$ the last equality of (9) gives the well known relation, (cf. [4] p. 33),

$$\sum_{r=0}^{p} S(n,r)(p)_r = p^n, \qquad (p)_r = p(p-1)\cdots(p-r+1),$$

Spacing on objects, equality condition. Denote by $R_k(n,\ p)$ the number of distributions of the n objects into the p unlike cells such that no cell contains objects whose difference is exactly k and denote by $T_k(n,\ p)$ the number of distributions counted in $R_k(n,\ p)$ but with the added condition that no cell is empty. Then using a similar argument such as that in obtaining the number $N_k(n,\ p)$ we have

$$R_k(n,p) = p^k(p-1)^{n-k} \tag{10}$$

and

$$T_k(n,p)=\sum_{i=0}^{p-1}(-1)^i\binom{p}{i}R_k(n,p-i)$$
$$=\sum_{i=0}^{p-1}(-1)^i\binom{p}{i}(p-i)^k\sum_{j=0}^{n-k}(-1)^j\binom{n-k}{j}(p-i)^{n-k-j}$$

and from (1) and (2),

$$=p!\sum_{j=0}^{n-k}(-1)^j\binom{n-k}{j})S(n-j,p). \tag{11}$$

From (10) and (11) the following summation is apparent,

$$\sum_{s=0}^{p-1}\binom{p}{s}T_k(n,p-s)=p^k(p-1)^{n-k} \tag{12}$$

and

$$\sum_{s=0}^{p-1}\binom{p}{s}\sum_{j=0}^{n-k}(-1)^j\binom{n-k}{j}S(n-j,p-s)(p-s)!=p^k(p-1)^{n-k}.$$

The relation

$$T_k(n,p)=pT_k(n-1,p-1)+(p-1)T_k(n-1,p) \tag{13}$$

is easily verified. Clearly, $N_2(n,p)=T_1(n,p)$ and hence from (5) and (11) follows, that

$$S(n-1,p-1)=\sum_{j=0}^{n-1}(-1)^j\binom{n-1}{j}S(n-j,p).$$

Spacing on cells. Define the spacing of any two cells c_i and c_j as $|i-j|$. We consider now distributions of the n unlike objects $a_1,\cdots,a_n$ into the p unlike cells $c_1,\cdots,c_p$ with added conditions on cell spacings. First we state the following easily established result (cf. [2], formula (35) or [3], expression (18)).

LEMMA. *The number of r-combinations* $x_1<x_2<\cdots<x_r$ *from* $\{1,2,\cdots,m\}$ *such that* $w\leqq x_{j+1}-x_j$, $j=1(1)r-1$ *and* $x_r-x_1\leqq w'$ *is*

$$C_{m,r}(w,w')=\frac{m-(r-1)(w+w'-m)}{r}\binom{w'-(w-1)(r-1)}{r-1}. \tag{14}$$

Denote by $H_k^{w,w'}(n,p)$ the number of distributions of the n unlike objects into the p unlike cells such that

(a) no two cells whose spacing is less than w both contain objects,
(b) no two cells whose spacing is greater than w' both contain objects, and
(c) no two objects whose spacing is less than k are both in the same cell.

It is easily seen, using (5) and (14), and with $\binom{n}{k}=0$ when n is negative, that

$$H_k^{w,w'}(n,p) = \sum_{r=1} N_k(n,r)C_{p,r}(w,w')$$

$$(15) \qquad = \sum_{r=1} r!\,S(n-k+1, r-k+1)\lfloor p-(r-1)(w+w'-p)\rfloor \binom{w'-(w-1)(r-1)}{r-1} \Big/ r.$$

The special case $H_1^{w,w'}(n, p) = \sum_{r=1} r! S(n, r) C_{p,r}(w, w')$ is the number of distributions with conditions (a) and (b) only. The number

$$(16) \quad H_k^{w,p-1}(n,p) = \sum_{r=1} r! S(n-k+1, r-k+1)\binom{p-(r-1)(w-1)}{r}$$

is the number of distributions with conditions (a) and (c) only.

Remarks. An ordered sequence

$$(17) \qquad b_1, b_2, \cdots, b_r, \qquad b_i \in \{1, 2, \cdots, m\}$$

is called an r-permutation, repetitions allowed, from m. The number of such permutations is equal to the number of ways of distributing r unlike objects $a_1, \cdots, a_r$ into m unlike cells $c_1, \cdots, c_m$ for, if object a_i is placed into cell c_j, let $b_i = j$ thus establishing a one-one correspondence between the r-permutations and the distributions. The number is, of course, m^r. Clearly, this number is also equal to the number of functions from a finite set of r objects into a finite set of m objects, while $N(n, p)$ is the number when the mappings are "onto."

Analagous to the previous restrictions with regard to spacings in the distributions we have the following results. $M_k(r, m)$ is the number of permutations (17) such that $b_i \neq b_j$ if $|i-j| < k$ and $i \neq j$ while $N_k(r, m)$ adds the extra condition that for any integer i, $i = 1, \cdots, m$ there exists at least one integer j such that $b_j = i$. Replacing the condition "$b_i \neq b_j$ if $|i-j| < k$" by "$b_i \neq b_j$ if $|i-j| = k$" in the preceding sentence we need to replace the numbers $M_k(r, m)$ and $N_k(r, m)$ by $R_k(n, p)$ and $T_k(n, p)$ respectively. $H_k^{w,w'}(r, m)$ is the number of r-permutations (17) such that for any two integers b_i, b_j, $i \neq j$, if $b_i \neq b_j$ then $w \leqq |b_i - b_j| \leqq w'$ and if $|i-j| < k$ then $b_i \neq b_j$. The associated probabilities in random sampling may be of interest. For example, a person draws r objects with replacement, order counting, from an urn containing m distinct objects. The probability that between any two successive draws of an object at least $k-1$ other draws have been made is $M_k(r, m)/m^r$, while $N_k(r, m)/m^r$ is the probability when the condition that every object is drawn at least once is added.

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TWO-DIMENSIONAL LATTICES AND CONVEX DOMAINS

SIMEON REICH, The Technion, Israel Institute of Technology, Haifa, Israel

Let A, P, d denote the area, the perimeter and the diameter respectively of a plane convex domain D. Three results have been established quite recently:

THEOREM 1 (Bender, [1]). *If $A > P/2$, then D contains a lattice point.*

THEOREM 2 (Hammer, [2]). *If $A > rP/2$, where r is any positive integer, then D contains r lattice points.*

THEOREM 3 (Hammer, [3]). *If $A \geqq 2^r P$, where r is any positive integer, then D contains $2^{r+2}-1$ lattice points.*

(Actually, the theorem holds even if $r=0$). To these we add:

THEOREM 4. *If $A > r(P/2+d)$, where r is any positive integer, then D contains $2r$ lattice points.*

Proof. By [4] there is a line which bisects both the area of D and its perimeter. We will call the two domains obtained in this way D_i, $i=1, 2$. Now $A_i = A/2 > \frac{1}{2}r(P/2+d) \geqq rP_i/2$. By Theorem 2, D_1, as well as D_2, contains r lattice points. This means that D contains $2r$ lattice points.

THEOREM 5. *If $A \geqq 2^r(P/2+d)$, where r is any positive integer, then D contains $2^{r+2}-2$ lattice points.*

Proof. Bisecting area and perimeter again, we have this time $A_i = A/2 \geqq 2^{r-1}(P/2+d) \geqq 2^{r-1}P_i$. By Theorem 3, the result follows.

REMARK. These theorems, and in particular Theorem 4, improve Theorems 2 and 3 because $d < P/2$. We present now two generalizations:

THEOREM 6. *If $A > r(P/2+(2^k-1)d)$, where k is a positive integer, then D contains $2^k r$ lattice points.*

Proof (by induction). The case $k=1$ is Theorem 4. Suppose now that the proposition is true for k. Then, after bisecting area and perimeter simultaneously we have $A_i = A/2 > r[(P/2+d)/2 + (2^{k+1}-2)d/2] = r[(P/2+d)/2+(2^k-1)d] > r(P_i/2+(2^k-1)d_i)$. Thus both D_1 and D_2 contain $2^k r$ lattice points. Therefore D contains $2^{k+1}r$ lattice points.

THEOREM 7. *If $A \geqq 2^r(P/2+(2^k-1)d)$, then D contains $2^k(2^{r+1}-1)$ lattice points.*

Proof (by induction). As before, the case $k=1$ being Theorem 5.
Other generalizations can be obtained in a similar fashion.

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4. A. C. Zitronenbaum, Bisecting an area and its boundary, Math. Gaz., 43 (1959) 130–131.

BOOK REVIEWS

EDITED BY D. ELIZABETH KENNEDY, University of Victoria

Materials intended for review should be sent to: Professor D. Elizabeth Kennedy, Department of Mathematics, University of Victoria, Victoria, British Columbia, Canada.

Reviews of texts at the freshman-sophomore level based upon classroom experience will be welcomed by the Book Review Editor.

A boldface capital **C** *in the margin indicates a classroom review.*

C *An Introduction to Matrices and Linear Transformations.* By John H. Staib. Addison-Wesley, Reading, Massachusetts, 1969. xii+336 pp. $8.95.

This linear algebra book is intended for use as a text primarily for sophomore science and engineering students. Any book designed for such a purpose faces a stiff challenge; it must appeal to a student who would much rather be in a laboratory "playing" with a laser beam than determining the kernel of a linear transformation, and yet it must present solid, somewhat abstract, mathematics. This reviewer is currently using this book for the second time for a class of such students and feels that it meets the task remarkably well.

The text is not written in the terse mathematical style which is usually appealing to the instructor and frustrating to the beginning student but rather in "a classroom style of presentation, with all its informality and experimentation." This rendered the book quite readable to the students and so essentially the entire book could be covered in the one semester. The students' background is assumed to include some calculus; familarity with the derivative and integral is needed in various of the examples and exercises.

The five chapters are entitled: Matrices; Vectors; Linear Transformations; The Determinant Function; Euclidean Spaces. Elementary row operations are introduced at the start and used throughout; in fact the book does a very good job of utilizing only a few basic techniques to take care of all the computational problems; even an alternative to the Gram-Schmidt process is presented (in an exercise) which uses only these fundamental techniques. (This procedure may also be found in an article by the book's author in the September 1969 issue of this MAGAZINE.) Quadratic forms are discussed early. Only vector spaces over the reals are considered (except in an exercise). Determinants are defined inductively and interpreted in terms of volume in an appendix. Characteristic values are discussed in the last two chapters. Several applications to geometry are clearly presented and a discussion of the use of matrices in solving linear differen-

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Misprints are few and minor and most are included in an accompanying list of errata. There are a good number of appropriate problems, answers for half of which are in the back of the book and for the rest in a separate booklet which may be obtained from the publisher. A few more difficult theoretical exercises might have been included to challenge the best students.

It is my opinion that this book serves as an excellent text for the sophomore course in linear algebra for the science or engineering student.

J. V. Michalowicz, The Catholic University of America

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Professor Lang states in the foreword to the First Course that "This book is written for the student, to give him an immediate, and pleasant, access to the subject. I hope that I have struck a proper compromise between dwelling too much on special details, and not giving enough technical exercises, . . . In any case, certain routine habits of sophisticated mathematicians are unsuitable for a first course."

One cannot complain about an author who writes expressly for the benefit of the student, or who attempts to make the subject as accessible and agreeable as possible. But "pleasant" seems to denote "easy" and it is always the case that learning mathematics, even in the beginning, requires hard work. It is the very "pleasantness" of the first part of the book which leaves the student unprepared to cope with the way integration theory is developed. In fact, Lang's development of this theory seems to be one of those "routine habits of sophisticated mathematicians" which he tried to avoid.

The book is roughly divided into two parts; the first comprises Chapters I–VIII and includes differentiation and the development of elementary functions; the second comprises Chapters IX–XVI and includes integration theory, technique, applications, and series. Emphasis is on the intuitive throughout the first part of the book. The concept of limit is regarded as intuitively clear and is used almost immediately with no formal definition and little motivation, in the definition of derivative. When needed, the limit theorems appear without proof, taken essentially as axioms. Moreover, continuity is not defined until the section on the mean value theorem (p. 93), and any further discussion of this concept, except for the statement that differentiable functions are continuous, is left until the chapter on integration. This passing over continuity is less than satisfying since the continuity hypothesis occurs in almost every theorem in the tial equations is given. The instructor must develop his own applications to the physical sciences, but this is probably as it should be, for these applications should be chosen carefully to meet the scientific interests and preparation of one's particular class.

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Given the above concessions, which would not be apparent to the student, the actual exposition of the first eight chapters is as "pleasant" as any student could hope. Arguments are brief, often area-geometric in nature, and definitions of elementary functions are chosen to give the most economical development of their properties. Students readily appreciate the flavor of these sections, and the author does achieve "immediacy" in the sense that the student, early in the course, can work with some nontrivial tools of calculus. A minor annoyance in this part of the book is the inclusion of the second derivative test for convexity in a section on graphing instead of the earlier section on the mean value theorem which included the first derivative test.

A real problem in teaching from this book arises in Chapter IX, Integration. Even here the first three sections are written so that the student, with his knapsack of "immediate" (but crude) tools, finds them easy to follow. Included in these sections is the concept of antiderivative and a proof that if $f(x)$ is a nonnegative function continuous on $[a, b]$, $F(x)=$ "area" under the graph of f from a to x, then $F'(x)=f(x)$ for x in (a, b). It is in the fourth section that Lang shifts gears, accelerates through the rest of the chapter, and leaves the bewildered student standing in his child's garden of calculus. For, from that point to the end of the chapter (approximately ten pages), the student is apparently expected to assimilate (i) an abstract formulation of the fundamental theorem of calculus, (ii) the concept of upper and lower sums with the attendant partition arguments, (iii) the concept of least upper bound and greatest lower bound (with no examples and no problems), and (iv) a section on integrable functions. All of this is couched in sophisticated and abstract notation with which the student has had no previous experience in the book. Moreover, in these ten pages there are precisely eight exercises given. After integration theory, the book proceeds through integration technique and applications. There is little to recommend in these chapters, for their development is standard, even shallow. Techniques of integration are, as a whole, lightly treated except for partial fractions, which receive about one third (in pages) of the emphasis. Moreover, in the applications chapter there are very few worked examples and, in fact, no problems for the sections on density and mass and on moments.

The last large segment of Lang's book deals with series or series-related topics. This consists basically of a chapter on Taylor's formula, and a chapter on series which includes a short treatment of power series. The material on Taylor's formula is motivated by a desire to numerically approximate values of the elementary functions. To this end, the standard remainder theorems are proved, and these remainders are found for the trigonometric functions, logarithm, etc. Nowhere in this chapter, or in the next on series, is the concept of

Taylor series mentioned, let alone uniform convergence of series of functions. The sections which give convergence tests for series contain only the most basic tests and test series.

Finally, a lack of quality, and sometimes quantity, in exercises is a recurring problem. The exercises are sometimes inappropriate, out of order (with respect to the body of text preceding them), and there are many mistakes in the answers printed at the end of the book. There is also a lack of good worked examples to illustrate concepts. In a book dedicated to giving students immediate and intuitive access to calculus, special attention should have been given to these details.

Note: The reviewer has had teaching experience with this book at the University of California, Los Angeles, at Occidental College, and at the University of Victoria. The book was used for a life science–social science course at the first two schools, and for the first year of a two year "mainstream" course at the last school.

C. R. MIERS, University of Victoria

An Introduction to Number Theory. By Harold M. Stark. Markam Publishing Company, Chicago, 1970. x+347 pp. $8.50.

Number theory is one of the few disciplines in higher education having demonstrable results whose history outdates the very idea of a university, an academy, or a lyceum. To sample the logical and historical stability of basic number-theoretic notions, we need only recall that, to this day, the three number theory books of Euclid's *Elements* not only survived the Dark Ages, but have never been subject to the logical "scandals," "gaps," and "errors" of his geometric books. Indeed, better than half-a-dozen of Euclid's theorems on number theory, suitably generalized or modified, are taught in every beginning course in higher arithmetic. Other Euclidean arithmetical theorems are mostly forgotten in modern curricula as numerological curiosities, since they were intended, in part, as mnemonics for quantifying notions in ethics (e.g., amicable numbers and square numbers) and aesthetics (e.g., perfect numbers). So much for the ancient traditions of number theory!

Elementary unique factorization is the single most prevalent theme pervading nearly every nook and cranny of this handy textbook intended for students who plan to terminate their formal instruction in mathematics. The book's author is well known for his part in settling a difficult theoretical problem on the cardinality of the set of *imaginary* quadratic fields with class number equal to one, i.e., whose integers enjoy unique factorization. So once again, an author on higher arithmetic is following H. Weyl's advice (*Algebraic Theory of Numbers*, Princeton University Press, 1940, pg. 70): *In summarizing, one may venture to say that K* [Kronecker's theory] *is the more fundamental, D* [Dedekind's theory] *is more complete; or that D is of higher importance to the geometer, who ought to be concerned about manifolds of every dimension, while K is more important to the arithmetician, whose chief concern (presuming he is old-fashioned enough!) is the law of unique factorization.*

Nearly three-quarters of this useful monograph is devoted to elementary multiplicative number theory (e.g., divisibility theory and the theory of con-

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Nearly three-quarters of this useful monograph is devoted to elementary multiplicative number theory (e.g., divisibility theory and the theory of congruences). The remainder of the book is devoted to the rational approximation of irrational algebraic numbers, the geometry of continued fractions, and magic squares, but ignores such standard topics as additive number theory, Riemann's zeta function (the exercise on page 50 for Euler's "proof" of the infinitude of the primes comes close to defining the zeta function), cyclotomy (e.g., the application of prime number theory to determining the necessary and sufficient conditions for a regular polygon to be constructible by ruler and compasses alone), Farey series, and use of the division algorithm for uniquely representing integers in "arbitrary" number bases. The latter item would be particularly enlightening to future teachers of "new math." Almost without exception, the author gives meticulous attention to every minute detail in statements and proofs. Six minor exceptions, which might confuse some novices, are the following: (1) On page 2 one finds, "We will see in Chapter 3 that no polynomial can give only primes." The statement and proof in Chapter 3 eliminate the identically constant polynomials. (2) On page 172 there is the statement, "He [Liouville] proved that real algebraic numbers cannot be too closely approximated by rational numbers." Actually, the theorem is formulated so that a rational number cannot be approximated (perfectly) by itself. (3) In the proof of the author's version of Liouville's approximation theorem on page 173, the fundamental theorem of algebra is used as a lemma. Ironically, the *uniqueness* of factorization into a complex constant times a product of monic linear factors is the result actually used in the estimates of β and r, rather than just the *existence* of such monic linear factors. (4) On page 93, it is proved that a polynomial congruence (mod p) has at *most* n distinct roots. It might be helpful to point out that the Chinese Remainder Theorem (page 72) provides a criterion for the *existence* of such a root when the modulus is arbitrary. Similarly, on page 332, a footnote could mention that the existence of a root field for the ground field F guarantees the existence of a zero for a polynomial over F. (5) In the list of symbols on page 344, M_n is listed. However, it does *not* refer to the Mersenne numbers M_n treated by the author. Further, F_n, the Fermat numbers discussed by the author are not listed on page 344. (6) On page 42, there is the comment, "Euclid knew the form of all even perfect numbers." Actually, the form of the even perfect numbers was settled jointly by Euclid and (some 2000 years later) Euler; one gave the "if" condition and the other the "only if" condition.

In this reviewer's opinion, the eighth and last chapter on quadratic fields is the best and most carefully written chapter in the book. It contains a wealth of excellent material on Euclidean domains and unique factorization domains. E.g., *the* 21 quadratic Euclidean domains are listed and discussed as well as *the* nine (*not* ten!) imaginary quadratic fields with the unique factorization property. To this day (May 12, 1970), the difficult problem of the cardinality of the set of all *real* quadratic fields with the unique factorization property remains open. No new insight into this outstanding problem is given. Various exercises of this chapter make reference to Dirichlet's theorem on primes in arithmetic progression and lead the student to interesting special cases of the quadratic reciprocity law. Surprisingly, one finds some space devoted to the sublime topic of Kummer's ideal numbers (whatever happened to Kronecker's constructive theory of ideals?), but no space is given to either the (standard) ideal theory of Dedekind or to the theory of finite abelian groups (which unifies the study of residue systems and generalizes the fundamental theorem of arithmetic).

The author's style is refreshingly lively, but, still logically concise. Humor and irony are used sparingly, and, then mostly in footnotes. E.g., on page 224, "Q.E.D." is translated from the Latin as "Quite Easily Done," when actually it stands for "Quite Elegantly Done!" (The "Q.E.F." of Euclidean constructions has to be treated more delicately.)

The work is far from being encyclopedic, nor is it intended to be such, that is, it does not approach the much needed "up-dating" of L. E. Dickson's three brilliant volumes on the History of Number Theory to include the last fifty years of exponential growth to number theory. Nearly 150 exercises of varying degrees of difficulty add greatly to the value of this fine book. All in all, the monograph is highly recommended for undergraduate mathematics majors and those in teacher-training courses.

A. A. MULLIN

PROBLEMS AND SOLUTIONS

EDITED BY ROBERT E. HORTON, Los Angeles Valley College

ASSOCIATE EDITOR, MURRAY S. KLAMKIN, Ford Scientific Laboratory, Dearborn, Michigan

Readers of this department are invited to submit for solution problems believed to be new that may arise in study, in research, or in extra-academic situations. Problems may be submitted from any branch of mathematics and ranging in subject content from that accessible to the talented high school student to problems challenging to the professional mathematician. Proposals should be accompanied by solutions, when available, and by any information that will assist the editor. Ordinarily, problems in well-known textbooks should not be submitted.

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Solutions should be submitted on separate, signed sheets. Figures should be drawn in India ink and exactly the size desired for reproduction.

Send all communications for this department to Robert E. Horton, Los Angeles Valley College, 5800 Fulton Avenue, Van Nuys, California 91401.

To be considered for publication, solutions should be mailed before December 1, 1970.

PROPOSALS

768. *Proposed by J. A. H. Hunter, Toronto, Canada.*

Solve the alphametic

$$\begin{array}{r} A \\ G\ O \\ G\ O \\ \underline{G\ A\ L} \\ L\ O\ O\ K \end{array}$$

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Solutions should be submitted on separate, signed sheets. Figures should be drawn in India ink and exactly the size desired for reproduction.

Send all communications for this department to Robert E. Horton, Los Angeles Valley College, 5800 Fulton Avenue, Van Nuys, California 91401.

To be considered for publication, solutions should be mailed before December 1, 1970.

PROPOSALS

768. *Proposed by J. A. H. Hunter, Toronto, Canada.*

Solve the alphametic

$$\begin{array}{r} A \\ G\;O \\ G\;O \\ \underline{G\;A\;L} \\ L\;O\;O\;K \end{array}$$

769. *Proposed by Charles W. Trigg, San Diego, California.*

A triangular number is composed of nine distinct digits in the decimal system. When it is sectioned into three triads, each triad is prime. Find the number and show it to be unique.

770. *Proposed by John E. Hafstrom, California State College at San Bernardino.*

Given $f(x)$ continuous on $[a, b]$. Prove that numbers c and d exist such that for $a<c<d<b$ we have

$$\frac{\int_a^b f(x)dx}{b-a} = \frac{\int_c^d f(x)dx}{d-c}.$$

771. *Proposed by Douglas Lind, Cambridge University, England.*

Show that the sum of the elements of a finite field of more than two elements must be zero.

772. *Proposed by Erwin Just, Bronx Community College.*

Let p be a prime and $\{w_i\}$, $i=1, 2, 3, \cdots p-1$, be the primitive pth roots of unity. If a set of rational numbers, $\{r_1, r_2, \cdots r_{p-1}\}$ is chosen so that $\sum_{i=1}^{p-1} r_i w_i$ is rational prove that $r_1=r_2= \cdots =r_{p-1}$.

773. *Proposed by Norman Schaumberger, Bronx Community College.*

Let M be an arbitrary point not necessarily in the plane of triangle $A_1A_2A_3$. If B_i is the midpoint of the side opposite A_i prove

$$\sum_{i=1}^{3} MA_i^2 - \sum_{i=1}^{3} MB_i^2 = 1/3 \sum_{i=1}^{3} A_iB_i^2.$$

774. *Proposed by A. W. Walker, Toronto, Canada.*

If (a, b, c) are the lengths of the sides of any triangle, show that:

$$3\left(\frac{a^2}{b^2}+\frac{b^2}{c^2}+\frac{c^2}{a^2}\right) - (a^2+b^2+c^2)\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right) \geqq 0.$$

SOLUTIONS

Late Solutions

C. T. Haskell, California State Polytechnic College, San Luis Obispo: **744**; *John Reigsecker, Chicago, Illinois:* **740**; *Henry J. Ricardo, Yeshiva University:* **740**; *J. Ernest Wilkins, Jr., Gulf General Atomic Incorporated, San Diego, California:* **744.**

Erratum

Quickie 411 [March, 1970], Page 111, fourth line from the bottom, "number" should be "numbers"; third line from the bottom, $6k+1$ should be $6k\pm 1$.

A Prime Game

747. [January, 1970] *Proposed by J. A. H. Hunter, Toronto, Canada.*

The usual conditions apply to this alphametic. We have a prime $G\ A\ M\ E$ here, so what is this $G\ A\ M\ E$?

$$
\begin{array}{cccc}
 & & & A \\
M & A & T & H \\
M & A & G' & S \\
\underline{M} & \underline{A} & \underline{T} & \underline{H} \\
G & A & M & E
\end{array}
$$

Solution by Santo M. Diano, Philadelphia, Pennsylvania.

M must be contained in the set $\{1, 2, 3\}$ and A in the set $\{4, 5, 9\}$ which implies that G is in $\{4, 7, 8\}$. Checking a table of primes gives 8923 as the only consistent choice for this $G\ A\ M\ E$.

The remaining numbers easily fall into place as:

$$
\begin{array}{cccccccccc}
 & & & 9 & & & & & 9 \\
2 & 9 & 6 & 7 & & 2 & 9 & 6 & 5 \\
2 & 9 & 8 & 0 & \text{or} & 2 & 9 & 8 & 4 \\
\underline{2} & \underline{9} & \underline{6} & \underline{7} & & \underline{2} & \underline{9} & \underline{6} & \underline{5} \\
8 & 9 & 2 & 3 & & 8 & 9 & 2 & 3
\end{array}
$$

Also solved by Merrill Barnebey, Wisconsin State University at LaCrosse; Haig Bohigian, John Jay College of Criminal Justice, New York; Richard L. Breisch, Pennsylvania State University; Martin J. Brown, Northern Community College, Covington, Kentucky; Thomas Cioppa, Long Island University; Edward A. Cygan, Jr., University of Illinois; Howard Darvas, Brooklyn, New York; Clayton W. Dodge, University of Maine; Joel Feingold, Brooklyn, New York; Harry M. Gehman, SUNY at Buffalo; Anton Glaser, Pennsylvania State University; M. G. Greening, University of New South Wales, Australia; George Gruber, Brooklyn, New York; Delores Haverstick, Fort Belvoir, Virginia; Philip Haverstick, Fort Belvoir, Virginia; Yul J. Inn, San Mateo, California; Birger Jansson, Research Institute of National Defense, Stockholm, Sweden; Alfred Kohler, Long Island University; Carl P. McCarty, La Salle College, Pennsylvania; Michael D. McKay, Alan H. Feweson and William A. Schmidt (jointly), Texas A and M University; Joseph V. Michalowicz, Catholic University of America; Henrietta O. Midonick, New York, New York; John W. Milsom, Butler County Community College, Pennsylvania; Thomas Moore, Kingston, Massachusetts; Prasert Na Nagara, Kasetsart University, Thailand; William Nuesslein, Sienna College, New York; J. F. Leetch, Bowling Green State University, Ohio; C. C. Oursler, Southern Illinois University at Edwardsville; Bernard J. Portz, S.J., Creighton University; Marilyn Rodeen, San Francisco, California; E. F. Schmeichel, College of Wooster, Ohio; Charles W. Trigg, San Diego, California; John R. Tucker, Washington College, Maryland; John R. Ventura, Naval Underwater Weapons Research and Engineering Station, Rhode Island; Allen Widrowicz, East Moline, Illinois; Kenneth M. Wilke, Topeka, Kansas; and the proposer.

A solution was received from *H. R. Leifer*, Pittsburgh, Pennsylvania, in which the G' was assumed to be a different number from G. Three prime solutions were found. $A\ M\ G\ G'\ T\ H\ S\ E$ equal respectively 5 2 7 1 0 3 8 9, 5 2 7 1 0 4 6 9 and 9 2 8 6 7 5 4 3. Three incorrect solutions were received.

Powers of Two

748. [January, 1970] *Proposed by Marlow Sholander, Case Western Reserve University.*

For $2x=b-a>0$ it is well known that $2^a+2^b>2^{b+1-x}$. Show that $2^a+2^b<2^{b+p}$ where $p=1/(1+2x^2)$.

Solution by Clayton W. Dodge, University of Maine.

If $y>0$, then $2^y>1$ and $2^{-y}<1$. Thus, since $b-a>0$ and $1/(1+2x^2)>0$, we have

$$1+2^{-(b-a)}<2<2^{1+1/(1+2x\ )}.$$

Multiplying the left and right sides by 2^b yields the desired inequality.

Also solved by E. F. Schmeichel, College of Wooster, Ohio; and the proposer.

A Relation Between Altitudes and Sides

749. [January, 1970] *Proposed by Simeon Reich, Israel Institute of Technology, Haifa, Israel.*

Let ABC be an acute-angled triangle. Prove that $\frac{1}{2}<(h_a+h_b+h_c)/(a+b+c)<1$, where h_a, h_b, h_c are the altitudes of the triangle and a, b, c its sides.

I. *Solution by Santo M. Diano, Philadelphia, Pennsylvania.*

The altitudes satisfy the relationships $h_a=b\sin C$, $h_b=c\sin A$ and $h_c=a\sin B$. Therefore we have $h_a+h_b+h_c=b\sin C+c\sin A+a\sin B<a+b+c$ which yields

$$\frac{h_a+h_b+h_c}{a+b+c}<1. \tag{1}$$

Next, if the orthocenter divides the altitudes respectively into the segments labeled as follows:

$$h_a=x+w$$
$$h_b=v+z$$
$$h_c=y+u,$$

we have the inequalities:

$$u+v+v+x>b$$
$$x+y+y+z>c$$
$$z+w+w+u>a.$$

Addition gives $2(h_a+h_b+h_c)>a+b+c$ or

$$\frac{h_a+h_b+h_c}{a+b+c}>\frac{1}{2}. \tag{2}$$

Combining (1) and (2) gives the desired result.

II. *Solution by Leon Bankoff, Los Angeles, California.*

If P is *any* point within an acute triangle ABC, $AP+BP+CP>(a+b+c)/2$. Substituting H (the orthocenter) for P, we obtain

$$h_a + h_b + h_c > AH + BH + CH > (a + b + c)/2.$$

Furthermore, it is known that $\sqrt{3}(a+b+c) \geqq 2(h_a+h_b+h_c)$. (See Problem E 1427, *American Mathematical Monthly*, (1961) 296.)

Hence the improved inequality,

$$1/2 < (h_a + h_b + h_c)/(a + b + c) < \sqrt{3}/2.$$

Also solved by Derrill J. Bordelon, U.S. Naval Underwater Weapons Research and Engineering Station, Rhode Island; L. Carlitz, Duke University; Huseyin Demir, Middle East Technical University, Ankara, Turkey; Dewey C. Duncan, Los Angeles, California; Michael Goldberg, Washington, D.C.; M. G. Greening, University of New South Wales, Australia; George Gruber, Brooklyn, New York; Philip Haverstick, Fort Belvoir, Virginia; Thomas C. Kilker, Allentown College, Center Valley, Pennsylvania; Murray S. Klamkin, Ford Scientific Laboratory, Michigan; Lew Kowarski, Morgan State College, Maryland; Herbert R. Leifer, Pittsburgh, Pennsylvania; Carl P. McCarty, La Salle College, Pennsylvania; Charles McCracken; Henrietta O. Midonick, New York, New York; Paul Moulton, Temple University; Prasert Na Nagara, Kasetsart University, Thailand; Marilyn Rodeen, San Francisco, California; Arthur S. Shieh, Kent State University, Ohio; E. F. Schmeichel, Itasca, Illinois; William Sergio, The Cooper Union, New York; I. H. Semester, Victoria, Canada; P. D. Thomas, Naval Oceanographic Office, Washington, D.C.; A. W. Walker, Toronto, Canada; Allen Widrowicz, East Moline, Illinois; and the proposer.

Parallelepiped Volume

750. [January, 1970] *Proposed by Charles W. Trigg, San Diego, California.*

Evaluate the determinant

$$\begin{vmatrix} a^2+b^2-c^2-d^2 & 2bc-2ad & 2bd+2ac \\ 2bc+2ad & a^2-b^2+c^2-d^2 & 2cd-2ab \\ 2bd-2ac & 2cd+2ab & a^2-b^2-c^2+d^2 \end{vmatrix}$$

I. *Solution by Nigel F. Nettheim, Bureau of the Census, Washington, D.C.*

The determinant equals the volume of the parallelepiped formed by the three vectors whose Cartesian coordinates are given by the rows of the matrix. The squared length of the first vector is $(a^2+b^2+c^2+d^2)^2$ and the squared length of each of the remaining vectors is the same by symmetry. The inner product between the first pair of vectors is 0 and the inner product between each of the remaining pairs is the same by symmetry. Therefore the parallelepiped is a cube with volume $\pm(a^2+b^2+c^2+d^2)^3$; since the coefficient of a^6 is ± 1, the positive sign must be chosen, so the determinant equals $(a^2+b^2+c^2+d^2)^3$.

By the same method more general results could be obtained. For example, let $s_n = x_1^2+x_2^2+ \cdots +x_n^2$, let I_n be the identity matrix of order n and let A_n be the matrix of order n with (i, j) element $(-1)^{i+j+1} x_i x_j$; then, for $n=1, 2, \cdots$ we have

$$\left| s_n I_n + 2A_n \right| = -\overset{n}{s_n}.$$

II. *Solution by E. E. Morrison, University of Aberdeen, Scotland.*

Let D denote the given determinant, and let

$$A = \begin{vmatrix} a & d & -c \\ -d & a & b \\ c & -b & a \end{vmatrix}.$$

It follows easily that

$$\operatorname{adj} A = \begin{vmatrix} a^2+b^2 & bc+ad & bd-ac \\ nc-ad & a^2+c^2 & ab+cd \\ bd+ac & cd-ab & a^2+d^2 \end{vmatrix}$$

and that $A \cdot \operatorname{adj} A =$

$$\begin{vmatrix} a(a^2+b^2-c^2-d^2) & 2a(bc+ad) & 2a(bd-ac) \\ 2a(bc-ad) & a(a^2+c^2-b^2-d^2) & 2a(ab+cd) \\ 2a(ac+bd) & 2a(cd-ab) & a(a^2+d^2-b^2-c^2) \end{vmatrix}$$

i.e., $A \cdot \operatorname{adj} A = a^3 \cdot D' = a^3 \cdot D$; by the elementary properties of determinants we have $\operatorname{adj} A = A^2$ so that $a^3 \cdot D = A^3$.

Since $A = a(a^2+b^2+c^2+d^2)$ it follows immediately that $D = (a^2+b^2+c^2+d^2)^3$.

III. *Solution by Birger Jansson, Research Institute of National Defense, Sweden.*

$$\Delta = \begin{vmatrix} a^2+b^2-c^2-d^2 & 2bc-2ad & 2bd+2ac \\ 2bc+2ad & a^2-b^2+c^2-d^2 & 2cd-2ab \\ 2bd-2ac & 2cd+2ab & a^2-b^2-c^2+d^2 \end{vmatrix}.$$

By multiplication of Δ and its transpose Δ_T we get $\Delta^2 = \Delta \cdot \Delta_T =$

$$\begin{vmatrix} (a^2+b^2+c^2+d^2)^2 & 0 & 0 \\ 0 & (a^2+b^2+c^2+d^2)^2 & 0 \\ 0 & 0 & (a^2+b^2+c^2+d^2)^2 \end{vmatrix}$$

and consequently

$$\Delta = (a^2+b^2+c^2+d^2)^3.$$

Also solved by Winifred Asprey, Vassar College, New York; J. C. Binz, Bern, Switzerland; L. Dale Black, Lakehead University, Port Arthur, Canada; Haig Bohigian, John Jay College of Criminal Justice, New York; Derrill J. Bordelon, U.S. Naval Underwater Weapons Research and Engineering Station, Rhode Island; Huseyin Demir, Middle East Technical University, Ankara, Turkey; Dewey C. Duncan, Los Angeles, California; Michael Goldberg, Washington, D.C.; M. G. Greening, University of New South Wales, Australia; Philip Haverstick, Fort Belvoir, Virginia; H. J. Hein-

bockel, Old Dominion University; Murray S. Klamkin, Ford Scientific Laboratory, Michigan; Herbert R. Leifer, Pittsburgh, Pennsylvania; Carl P. McCarty, La Salle College, Pennsylvania; Joseph V. Michalowicz, Catholic University of America; Prasert Na Nagara, Kasetsart University, Thailand; E. F. Schmeichel, College of Wooster, Ohio; William O. Schmidt, Texas A and M University; Romesh Singh, Arthur District High School, Ontario, Canada; Marlow Sholander, Case Western Reserve University; Paul D. Thomas, Naval Oceanographic Office, Washington, D.C.; A. W. Walker, Toronto, Canada; Allen Widrowicz, East Moline, Illinois; Kenneth M. Wilke, Topeka, Kansas; Gregory Wulczyn, Bucknell University, Pennsylvania; and the proposer.

A Discontinuous Transformation

751. [January, 1970] *Proposed by Zalman Usiskin, University of Michigan.*

Prove or disprove: If a transformation of the plane maps each connected set onto a connected set, then the transformation is continuous.

Solution by Benjamin L. Schwartz, McLean, Virginia.

The conjecture is false. Consider the following function:

$$\begin{cases} f(x, y) = (\sin 1/x, y) \quad \text{for} \quad x \neq 0, \\ f(0, y) = (0, y). \end{cases}$$

This maps the plane into the vertical strip $\{(x, y) \mid -1 \leqq x \leqq 1$. It is discontinuous at every point on the line $x=0$, since $\lim_{x\to 0} f(x, y)$ does not exist. But it is easy to verify that connected sets are carried into connected sets, whether or not they include points where $x=0$.

Also solved by Robert E. Clay, Notre Dame University; Donald W. Hadwin, Augustana College, South Dakota; Douglas Lind, Stanford University; Joseph V. Michalowicz, Catholic University of America; William Nuesslein, Sienna College, New York; F. J. Papp, University of Lethbridge, Canada; G. P. Speck, Bradley University; and Simeon Reich, Israel Institute of Technology, Haifa, Israel.

F. J. Papp gave the reference: *Pervin* and *Levine*, *Connected Mappings in Hausdorff Spaces*, Proceedings of the AMS, 1, 1958, Pages 488–496.

A Trigonometric Series

752. [January, 1970] *Proposed by Norman Schaumberger, Bronx Community College.*

Prove that

$$\lim_{n\to 0} \sum_{k=1}^{n-1} \left(\tan \frac{k\pi}{2n} / n\right)^p$$

converges for $p > 1$ and diverges for $p \leqq 1$.

Solution by E. F. Schmeichel, College of Wooster, Ohio.

Set

$$S(n, p) = \sum_{k=1}^{n-1} \left(\frac{1}{n} \tan \frac{k\pi}{2n}\right)^p = \sum_{k=1}^{n-1} \left(\frac{1}{n} \cot \frac{k\pi}{2n}\right)^p.$$

For $0 < \theta < \pi/2$, $(1/n) \cot(\theta/n) = (\theta/n)/\theta \tan(\theta/n) < 1/\theta$ whence

$$\frac{1}{n}\cot\frac{k\pi}{2n} < \frac{2}{k\pi} \quad \text{and} \quad \lim_{n\to\infty}\frac{1}{n}\cot\frac{k\pi}{2n} = \frac{2}{k\pi}\cdot$$

So if $p>1$, a standard result (T. J. I.'a Bromwich, *Infinite Series*, Art. 49) yields

$$\lim_{n\to\infty} S(n, p) = \sum_{k=1}^{\infty}\lim_{n\to\infty}\left(\frac{1}{n}\cot\frac{k\pi}{2n}\right)^p = \sum_{k=1}^{\infty}\left(\frac{2}{k\pi}\right)^p,$$

which converges. On the other hand,

$$\frac{1}{k\pi} < \frac{1}{n}\cot\frac{k\pi}{2n}$$

for n sufficiently large, and so $\lim_{n\to\infty} S(n, p) \geqq \sum_{k=1}^{\infty} (1/k\pi)^p$, which diverges for $p \leqq 1$.

Also solved by David Gootkind, AVCO Computer Services, Wilmington, Massachusetts; and the proposer.

A Generator of Primes

753. [January, 1970] *Proposed by Michael J. Martino, Temple University, and John DeJoice, Aries Corporation, McLean, Virginia.*

Define $f(n) = C(n) - C(n-1)$ for all $n \geqq 2$ where

$$C(n) = \sum_{i=1}^{[\sqrt{n}]} [(n/i) - (i-1)].$$

(The brackets indicate the greatest integer.)

Show that $f(n) = 1$ implies n is prime.

Solution by L. Carlitz, Duke University.

Let $\tau(n)$ denote the number of divisors of n. Then (see for example Landau's *Vorlesungen über Zahlentheorie*, Vol. 2, p. 194)

$$\sum_{k\leq n}\tau(k) = 2\sum_{k\leq\sqrt{n}}\left[\frac{x}{n}\right] - [\sqrt{x}]^2.$$

Then

$$2C(n) = 2\sum_{k\leq\sqrt{n}}\left(\left[\frac{n}{k}\right] - k + 1\right) = 2\sum_{k\leq\sqrt{n}}\left[\frac{n}{k}\right] - [\sqrt{n}]([\sqrt{n}] - 1)$$

$$= \sum_{k\leq n}\tau(k) + (\sqrt{n}),$$

so that

$$2f(n) = \tau(n) + [\sqrt{n}] - [\sqrt{n-1}).$$

Hence, if $n = m^2$,

$$f(m^2) = \tfrac{1}{2}(\tau(m^2) + 1),$$

which if n is not a square

$$f(n) = \tfrac{1}{2}\tau(n).$$

It follows from this that $f(n) = 1$ if and only if n is prime.

It follows also that $f(n) = 2$ if and only if $n = p^2$ or pq, where p, q are distinct primes.

Also solved by M. G. Greening, University of New South Wales, Australia; Philip Haverstick, Fort Belvoir, Virginia; Dean Hickerson, Davis, California; Yul J. Inn, Aragon High School, San Mateo, California; Henry S. Lieberman, Waban, Massachusetts; Carl P. McCarty, La Salle College, Pennsylvania; Marilyn Rodeen, San Francisco, California; E. F. Schmeichel, College of Wooster, Ohio; Kenneth M. Wilke, Topeka, Kansas; Gregory Wulczyn, Bucknell University, Pennsylvania; and the proposers.

Intersections of Great Circles

130. [March, 1952] *Proposed by Leo Moser, University of Alberta.*

Prove that if n great circles on a sphere intersect in more than two points then they intersect in at least $2n$ points.

Solution by Leon Bankoff, Los Angeles, California.

If no more than two circles lie on a point, the number of intersections of n great circles on a sphere will be a maximum. For each combination of two circles there are two points of intersection so that we have a maximum of $2(n/2)$, or $n(n-1)$ intersections.

By rotating one of the circles until it passes through an existing intersection, the total number of intersections is reduced. If this process is repeated, successive reductions occur in the number of intersections until finally n circles lie on two poles.

When $(n-1)$ circles intersect in two points, the remaining circle provides $2(n-1)$ intersections, so that there is a total of $2(n-1)+2$, or $2n$ intersections. Further reduction in the number of intersections, effected by rotating the non-meridian circle until it becomes a meridian, results in n great circles intersecting in two points.

Comment by the proposer: The result is an immediate corollary (using duality and the relation between the sphere and projective plane) of the following: "Given n points which do not all lie on the same straight line, prove that if we join every two of them we obtain at least n distinct straight lines." [P. Erdös and Robert Steinberg, Problem 4065, *American Mathematical Monthly*, 51 (March, 1944), 169–170] H. S. M. Coxeter discussed a closely related problem in *A problem of collinear points, American Mathematical Monthly*, 55 (January, 1948), 26–28.

Comment on Problem 737

737. [September, 1969, and March, 1970] *Proposed by Irving A. Dodes, Kingsborough Community College, Brooklyn, New York.*

Mr. S. D. S. Hippie, inveterate seeker after truth (far after), awoke from a slight doze to hear his plane geometry teacher remark that if the midpoints of any quadrilateral are joined, the result is a parallelogram. Not to be outdone in untenable hypotheses, Mr. Hippie opined that if trisection points are joined, the result will also be a parallelogram.

Prove that no other dividing points independent of the lengths of the sides exist to produce a parallelogram.

Comment by Charles W. Trigg, San Diego, California.

Proceeding clockwise around the quadrilateral with vertices (x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4), the points that divide the sides in the ratio r are:

$$A[(x_2 - x_1)r + x_1,\ (y_2 - y_1)r + y_1],$$
$$B[(x_3 - x_2)r + x_2,\ (y_3 - y_2)r + y_2],$$
$$C[(x_4 - x_3)r + x_3,\ (y_4 - y_3)r + y_3],$$
$$D[(x_1 - x_4)r + x_4,\ (y_1 - y_4)r + y_4].$$

If the slopes of AB and CD are the same, then

$$\frac{(y_3 - 2y_2 + y_1)r + y_2 - y_1}{(x_3 - 2x_2 + x_1)r + x_2 - x_1} \equiv \frac{[(y_1 - 2y_4 + y_3)r + y_4 - y_3]k}{[(x_1 - 2x_4 + x_3)r + x_4 - x_3]k}, \qquad k \neq 0.$$

If this identity is to hold regardless of the lengths of the sides of the quadrilateral, then the absolute values of the numerators are equal, as are those of the denominators. Thus

$$r = \frac{y_1 - y_2 - k(y_3 - y_4)}{y_3 - 2y_2 + y_1 - k(y_1 - 2y_4 + y_3)},$$

or

$$r = \frac{y_1 - y_2 + k(y_3 - y_4)}{y_3 - 2y_2 + y_1 + k(y_1 - 2y_4 + y_3)}.$$

For the value of r to be independent of the lengths of the sides, k must be 1 (this is equivalent to saying that $AB = CD$) and

$$r = (y_1 - y_2 - y_3 + y_4)/2(y_4 - y_2) = (y_1 - y_3)/2(y_4 - y_2) + \tfrac{1}{2},$$

which varies with the positions of the vertices; or

$$r = (y_1 - y_2 + y_3 - y_4)/2(y_1 - y_2 + y_3 - y_4) = \tfrac{1}{2}.$$

Therefore, in general, the midpoints of the consecutive sides of the quadrilateral are the only ones which will always produce a parallelogram. Midpoints of the alternate sides of a quadrilateral determine a crossed quadrilateral.

Being words with more than four letters, "quadrilateral" and "rectangle" were very confusing to a half-awake Hippie. Since his mental image was that of a rectangle, his off-the-cuff (or, did he wear a shirt?) opinion was correct as it would have been for any corresponding n-section points on the sides of a rectangle, or for that matter, of a parallelogram. Indeed, he possibly saw that the joins of corresponding n-sectioning points on the sides of a square form a square, but did not so opine since that geometric term is anathema to him.

Constitutionally averse to doing anything in a conventional orderly fashion, Hippie's only consistency is his inconsistency. Trying to go two ways at the same time, his subconscious may have connected the two closest trisection points of a general quadrilateral at one vertex, then moved to the most remote point on the next adjacent side, then to the closest point on the next side, and then connected that to the starting point. In that event, after drawing the diagonal from the chosen vertex, he had two pairs of similar triangles each with a 2/3 ratio of similitude and each with a pair of parallel sides, one of them being the diagonal. Thus, the joins formed a parallelogram, as pointed out in the solution published on page 109 of the March, 1970, issue.

Hippie did state "if the trisection points are joined" without specifying how they were to be chosen. Since there are two on each side of a general quadrilateral, on consecutive sides they could be chosen in 2^4 ways only two of which yield parallelograms. On alternate sides, they could be selected and joined in $2(2^4)$ ways to produce crossed quadrilaterals. Thus, characteristically, Hippie's opinion was correct in only 1/24 of the cases his statement permitted. In the n-sectioned situation, only $(n-1)$ out of $3(n-1)^4$ or 1 out of $3(n-1)^3$ of the formable quadrilaterals are parallelograms.

Comment on Q470

Q470. Show that $2^{5n+1}+5^{n+2}$ is divisible by 27 for $n=0, 1, 2, \cdots$

[*Submitted by Alan Sutcliffe*]

Comment by Sid Spital, California State College at Hayward.

The published answer is in error (for example, try $n=0$). The correction should read

$$2^{5n+1} + 5^{n+2} = 2(27 + 5)^n + 5^n(27 - 2) = 27k.$$

QUICKIES

From time to time this department will publish problems which may be solved by laborious methods, but which with the proper insight may be disposed of with dispatch. Readers are urged to submit their favorite problems of this type, together with the elegant solution and the source, if known.

Q482. For any real numbers $a_i>0$ and any integers M, $P>0$ show that

$$M\left(\sum_{i=1}^{M} a_i^{P}\right) \leqq \left(\sum_{i=1}^{M} a_i^{P+1}\right)\left(\sum_{i=1}^{M} a_i^{-1}\right).$$

[*Submitted by D. E. Daykin, Reading, England*]

Q483. If A, B, C are the angles of a triangle such that

$$\tan(A - B) + \tan(B - C) + \tan(C - A) = 0$$

then the triangle is isosceles.

[*Submitted by Murray S. Klamkin*]

Q484. Show that the length of one leg of a Pythagorean triangle must be a multiple of 3.

[*Submitted by Charles W. Trigg*]

Q485. If concentric squares with parallel sides have areas in the ratio 2:1, then the segments drawn through the vertices of the smaller square perpendicular to the diagonals form with the segments of the larger square a regular octagon.

[*Submitted by Torquist Memp*]

Q486. Prove that for all integers n, $(n^5 - n)$ is divisible by 30.

[*Submitted by Robert S. Hatcher*]

(Answers on pp. 185–186.)

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the orthocenters of the other two. Then $\overline{YX}$ and $\overline{WZ}$ are perpendicular, and $YX/WZ = r$.

Proof. Let M be the midpoint of the diagonal $\overline{AC}$. If the triangles lie towards the quadrilateral's exterior, then $M^r M_{90}$ takes X to Z and Y to W. If the triangles lie toward the interior, apply $M^{1/r}M_{90}$ instead.

Again, the quadrilateral need not be convex or simple, and there is an analogous theorem for triangles.

Acknowledgements: I first saw dynamic proofs of Corollaries 1 and 2 in a course that Nicolaas Kuiper gave at the University of Michigan in 1953. Later that year Kenneth Leisenring showed me Theorem 3, and in 1965 he encouraged me to develop dynamic proofs for the African Mathematics Secondary Program. I am grateful to them.

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ANSWERS

A482. For any reals α, β the function

$$f(x) = \left(\sum_{i=1}^{M} x_i^{x-\alpha}\right)\left(\sum_{i=1}^{M} a_i^{\beta-x}\right)$$

is convex and symmetric about $x = 1/2(\alpha+\beta)$, and so increases as x moves away from $x = 1/2(\alpha+\beta)$. With $\alpha = 0$ and $\beta = P$ the given inequality is $f(P) \leqq f(P+1)$.

A483. Expanding out and replacing tan A, tan B and tan C by a, b, and c, respectively we get

$$\frac{a-b}{1+ab} + \frac{b-c}{1+bc} + \frac{c-a}{1+ca} = 0.$$

On combining fractions and factoring we obtain

$$(a-b)(b-c)(c-a) = 0$$

and thus the triangle is isosceles. Note the condition that $A+B+C=\pi$ is not necessary.

A484. Fermat's theorem states that if p is a prime and m is not divisible by p, then $m^{p-1} \equiv 1 \pmod{p}$. The relationship $a^2+b^2=c^2$ may be written as

$$(a^{3-1} - 1) + (b^{3-1} - 1) = c^2 - 2 = (c^{3-1} - 1) - 1.$$

If neither a nor b is divisible by 3, then each of the terms on the left-hand side of the equation is divisible by 3. But, whether c is divisible by 3 or not, the right-hand side of the equation leaves a remainder upon division by 3. Hence, for the equation to hold, one of a, b must be a multiple of 3.

A485. The difference of the half diagonals is $1-\sqrt{2}/2$, so each segment through the vertices is $2-\sqrt{2}$. At each corner the segments cut off from the sides are $(2-\sqrt{2})/\sqrt{2}$, so the remaining segment is $\sqrt{2}-2(2-\sqrt{2})/\sqrt{2}$ or $2-\sqrt{2}$. Each interior angle is 135° so the octagon is regular.

A486. An integer and its fifth power end in the same digit, hence (n^5-n) ends in zero and so is divisible by 2 and 5. Its factors are $(n-1)(n)(n+1)(n^2+1)$ of which one of the first three must be divisible by 3. So (n^5-n) must be divisible by $2\times 3\times 5$.

(Quickies on pp. 235–236.)

AN IMPLICATION OF THE PYTHAGOREAN THEOREM

JOHN Q. JORDAN and JOHN M. O'MALLEY, JR., Boston State College

It is well known that a Banach space in which the parallelogram law

$$\|x+y\|^2+\|x-y\|^2=2(\|x\|^2+\|y\|^2)$$

holds is a Hilbert space [**3**, p. 23]. This law is equivalent to the Pythagorean theorem. Also Hilbert space is a generalization of Euclidean space. Therefore it is natural to conjecture that the Pythagorean theorem implies Euclid's parallel postulate, or an equivalent proposition.

This conjecture is correct, but simple proofs of this fact do not seem to be well known. Two such proofs are presented in this note. The first is a straightforward application of the Pythagorean theorem in a general right triangle. The second proof is simpler, but rather artificial because it uses an isosceles right triangle rather than a general one.

The reader will recall that an absolute geometry is Euclidean if and only if there exists at least one triangle the sum of the measures of whose angles is 180 [**1**, p. 11]. We shall use this proposition rather than one of the more common forms (e.g., Playfair's) of Euclid's parallel postulate.

In absolute geometry if the Pythagorean theorem holds, it is trivially true that its converse is also valid.

After these preliminaries we can proceed with our proofs that the Pythagorean theorem implies Euclid's parallel postulate. We shall use two lemmas, one of which may be quite familiar to the reader.

LEMMA 1. *If a rectangle exists, then there is a triangle the sum of the measures of whose angles is* 180.

Proof. Given a rectangle $\square ABCD$. The sum of the measures of its angles is 360. Also the sum of the measures of these angles is the sum of the measures of

Q483. If A, B, C are the angles of a triangle such that

$$\tan(A-B)+\tan(B-C)+\tan(C-A)=0$$

then the triangle is isosceles.

[*Submitted by Murray S. Klamkin*]

Q484. Show that the length of one leg of a Pythagorean triangle must be a multiple of 3.

[*Submitted by Charles W. Trigg*]

Q485. If concentric squares with parallel sides have areas in the ratio 2:1, then the segments drawn through the vertices of the smaller square perpendicular to the diagonals form with the segments of the larger square a regular octagon.

[*Submitted by Torquist Memp*]

Q486. Prove that for all integers n, (n^5-n) is divisible by 30.

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